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Estimation Methods: Inference Classical and Bayesian of Extended Inverse Exponential Distribution

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ABSTRACT

The parameters of the Burr XII-Inverse Exponential (BXII-IE) distribution were estimated in this work using a variety of Bayesian and non-Bayesian (Inference Classical) approaches. Methods like Maximum likelihood, least squares, weighted least squares, Maximum product space estimators, and Anderson Darling were developed for non-Bayesian estimators. Due to the lack of closed-form solutions for Bayesian estimates for certain loss functions squared error (GE), general entropy (GE), linear-exponential (LN), and prior distributions for the parameters, Bayesian estimators had to be implemented. Bayesian estimation utilizing the Markov Chain Monte Carlo (MCMC) approach and seven non-Bayesian estimate techniques were tested for performance. The results indicated that, in the context of classical estimation, Anderson Darling and Maximum Likelihood Estimation are the most efficient estimators. Furthermore, Bayesian estimations utilizing the LN-0.7 and GE-0.7 loss functions demonstrated superior performance compared to their counterparts. Lastly, we use a single real-world COVID-19 dataset from Canada and apply the suggested OGRIW distribution. Compared to previous distributions, the new one is far more flexible, as shown by the outcomes of this application.

1. Introduction

In recent years, statisticians have increasingly focused on developing new generalized distributions rather than baseline models for research purposes. These generalized distributions provide new avenues for exploring real-world issues and offer more modeling flexibility for asymmetric and intricate random processes. As a result, the corpus of academic literature contains several models.

A large number of statisticians have introduced novel distributions such as the [1-21]

We examined Bayesian and non-Bayesian approaches for estimating the parameters of the BXII-E distribution. In the realm of statistical inference and data modeling, parameter estimation is crucial. In ambiguous situations, Bayesian methods provide adaptability and robustness by incorporating previous knowledge into parameter estimations. Non-Bayesian methods provide answers that are both computationally efficient and unequivocal. Numerous authors in this domain have examined and investigated parameter estimation methodologies, as referenced in [22-26].

The subsequent sections of the article are delineated as follows: Section 2 delineates the BXII-IE distribution. Section 3 addresses the methodologies of classical inference. Section 4 explores Bayesian estimation methodologies that use several loss functions, including Squared Error (SE), LINEX, and Generalized Error (GE), under the assumption of independent priors. Section 5 illustrates a hypothetical activity to

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demonstrate the distribution's flexibility, section 6 evaluates COVID-19 data analysis in real life, while Section 7 contains the Discussion.

2. The Odd Burr XII Inverse Exponential (BXII-IE) Distribution

We have developed this BXII-G family of distributions by the work of Khalaf [27].

$$Q(x)_{BXII-G} = 1 - \left[1 + \left[\frac{[F(x;\xi)]^2}{1-F(x;\xi)}\right]^a\right]^{-b} \quad (1)$$

The PDF of the BXII –G Family is

$$q(x)_{BXII-G} = baF(x;\xi)f(x;\xi)\left(2 - F(x;\xi)\right)$$
$$\times\left(1 - F(x;\xi)\right)^{-2}\left[\frac{[F(x;\xi)]^2}{1-F(x;\xi)}\right]^{a-1}\left[1 + \left[\frac{[F(x;\xi)]^2}{1-F(x;\xi)}\right]^a\right]^{-(b+1)} \quad (2)$$

The CDF and PDF of an Inverse Exponential (IE) distribution are defined as follows:[28]

$$F(x,c) = e^{-\frac{c}{x}} \quad (3)$$

and

$$f(x,c) = cx^{-2}e^{-\frac{c}{x}} \quad ,x > 0, c > 0 \quad (4)$$

Equation (1) is modified to create a novel distribution called The Odd Burr XII Inverse Exponential (BXII-IE) distribution. The equation that results from this substitution is:

$$Q(x)_{BXII-IE} = 1 - \left[1 + \left[\frac{e^{-2\frac{c}{x}}}{1-e^{-\frac{c}{x}}}\right]^a\right]^{-b} \quad (5)$$

The PDF of the BXII-IE distribution may be obtained by substituting equations (3) and (4) into equation (2):

$$q(x)_{BXII-IE} = bacx^{-2}e^{-2\frac{c}{x}}\left(2 - e^{-\frac{c}{x}}\right)$$
$$\times\left(1 - e^{-\frac{c}{x}}\right)^{-2}\left[\frac{e^{-2\frac{c}{x}}}{1-e^{-\frac{c}{x}}}\right]^{a-1}\left[1 + \left[\frac{e^{-2\frac{c}{x}}}{1-e^{-\frac{c}{x}}}\right]^a\right]^{-(b+1)} \quad (6)$$

where $x > 0, b, a, c > 0$

The hazard function for the BXII-IE is:

$$h(x)_{BXII-IE} = bacx^{-2}e^{-2\frac{c}{x}}\left(2 - e^{-\frac{c}{x}}\right)$$
$$\times\left(1 - e^{-\frac{c}{x}}\right)^{-2}\left[\frac{e^{-2\frac{c}{x}}}{1-e^{-\frac{c}{x}}}\right]^{a-1}\left[1 + \left[\frac{e^{-2\frac{c}{x}}}{1-e^{-\frac{c}{x}}}\right]^a\right]^{-b} \quad (7)$$

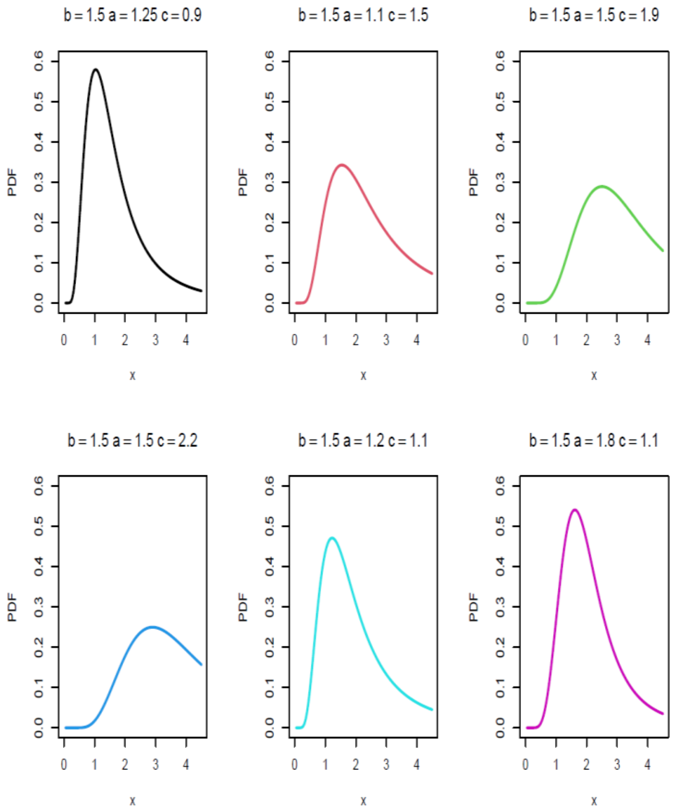

Figure 1. The plot pdf for the BXII-IE distribution

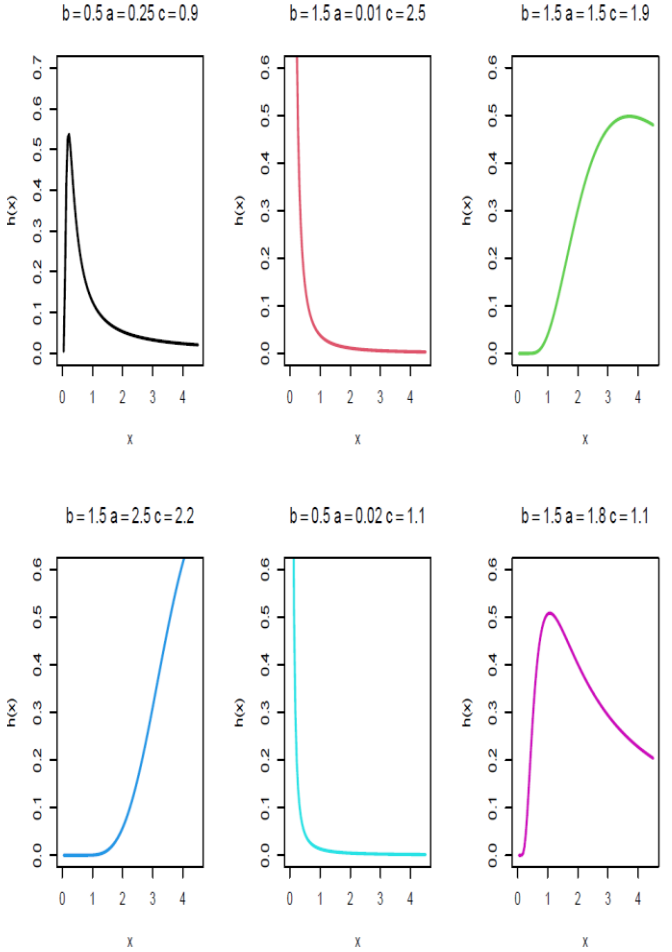

Figure 2. The plot hazard function for the BXII-IE distribution

3. Non-Bayesian Estimation of the BXII-IE distribution

3.1 Maximum Likelihood Estimation

According to references [29-32], the MLE approach is used to estimate b, a, and c by maximizing the log-likelihood function.

$$l(\varphi) = nlog(b) + nlog(a) + nlog(c)$$
$$+log\sum_{i=1}^{n} x_i^{-2} - 2\sum_{i=1}^{n}\frac{c}{x_i} + log\sum_{i=1}^{n}\left(2 - e^{-\frac{c}{x_i}}\right)$$
$$+(a-1)log\sum_{i=1}^{n}\left(\frac{e^{-2\frac{c}{x_i}}}{1-e^{-\frac{c}{x_i}}}\right)$$
$$-blog\sum_{i=1}^{n}\left(1+\left[\frac{e^{-2\frac{c}{x_i}}}{1-e^{-\frac{c}{x_i}}}\right]^{a}\right) \quad (8)$$

3.2 Least Squares Estimation (LSE)

See [33] to minimize its function.

$$L(\mathrm{x_i}) = \sum_{i=1}^{n}\left(F(x_i) - \frac{i}{n+1}\right)^2 = \sum_{i=1}^{n}\left(e^{-\frac{c}{x_i}} - \frac{i}{n+1}\right)^2 \quad (9)$$

3.3 Weighted Least Squares Estimation (WLSE)

WLSE (see to [35]) maximizes its function as follows:

$$W(\mathrm{x_i}) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(x_i) - \frac{i}{n+1}\right)^2 = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(e^{-\frac{c}{x_i}} - \frac{i}{n+1}\right)^2 \quad (10)$$

3.4 Maximum Product Space Estimators (MPSE)

It minimizes the MPSE function to estimate b, a, and c; for more information on MPSE, see [36].

$$S(\mathrm{x_i}) = \frac{1}{n+1}\sum_{i=1}^{n}\ln\left(F(x_i) - F(x_{i-1})\right) = \frac{1}{n+1}\sum_{i=1}^{n}\ln\left(e^{-\frac{c}{x_i}} - e^{-\frac{c}{x_{i-1}}}\right) \quad (11)$$

3.5 Anderson-Darling Estimation (ADE)

In [37], b, a, and c are estimated by minimizing their function.

$$D(\mathrm{x_i}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\left(\ln\left(F(x_i)\right) + \ln\left(S(x_i)\right)\right)$$
$$= -n - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\left(\ln\left(e^{-\frac{c}{x_i}}\right) + \ln\left(1 - e^{-\frac{c}{x_i}}\right)\right) \quad (12)$$

4. Bayesian Estimation of BXII-IE Distribution Parameters

In this section, we explain the Bayesian estimate (BE) for the unknown parameters of the BXII-IE distribution. Bayesian parameter estimation may take into account several loss functions, as pointed out by Albert [38] and Mood [39], including squared error loss (SEL), LINEX, and generalized entropy loss functions (GE). Separate gamma and beta priors might be considered for the b, a, and c variables. According to [40], previous distributions for the BXII-IE parameters and their pdfs were used.

$$\pi_1(b) \propto b^{s_1-1}e^{-p_1 b} \quad b, s_1, p_1 > 0 \quad (13)$$

$$\pi_2(a) \propto (a-1)^{s_2-1}e^{-p_2(a-1)} \; a > 1,, s_2, p_2 > 0 \quad (14)$$

$$\pi_3(\,c) \propto \; c^{s_3-1}e^{-p_3 c} \quad c, s_3, p_3 > 0 \quad (15)$$

hyper-parameters that describe what is known about the unknown parameters are selected using s_j and p_j, where j = 1, 2, and 3. The joint that comes before the function $\varphi = (a, b, c)$ is shown here.

$$\pi(\varphi) = \pi_1(b)\pi_1(a)\pi_1(c) \quad (16)$$

Substituted equations (13), (14), and (15) into (16) we get:

$$\pi(\varphi) \propto b^{s_1-1}(a-1)^{s_2-1}\, c^{s_3-1}e^{-p_1 b - p_2(a-1) - p_3\, c} \quad (17)$$

The joint posterior of ψ is expressed as follows:

$$\pi(\varphi \backslash x = x_1, x_2, \ldots, x_n) = \frac{\pi(\varphi)l(\varphi)}{\int_0^\infty \int_0^\infty \pi(\varphi)l(\varphi)d\varphi} \quad (18)$$

So

$$\pi(\varphi\backslash x) = b^{s_1}(a-1)^{s_2}\, c^{s_3}e^{-p_1 b - p_2(a-1) - p_3\, c}\prod_{i=1}^{n}\left(x^{-2}e^{-2\frac{c}{x}}\left(2 - e^{-\frac{c}{x}}\right)\left(1 - e^{-\frac{c}{x}}\right)^{-2}\left[\frac{e^{-2\frac{c}{x}}}{1-e^{-\frac{c}{x}}}\right]^{a-1}\left[1 + \left[\frac{e^{-2\frac{c}{x}}}{1-e^{-\frac{c}{x}}}\right]^{a}\right]^{-(b+1)}\right) \quad (19)$$

The following is a definition of the SE loss function, which was one of three loss functions examined in this study:

$$L_{SE}\left(l(\varphi), \hat{l}(\varphi)\right) = \left(l(\varphi) - \hat{l}(\varphi)\right)^2 \quad (20)$$

where the function , $\hat{l}(\varphi)$ is used to estimate $l(\varphi)$. And in a Bayesian context, the penalty for under- and over-estimation is the same when using the SE loss function. A workaround is to apply the LINEX and GE loss functions. In [41]. The LINEX loss function is defined in the following way:

$$L_{\text{LINEX}}\left(l(\varphi),\hat{l}(\varphi)\right)=e^{\rho\left(l(\varphi)-\hat{l}(\varphi)\right)} -\rho\left(l(\varphi),\hat{l}(\varphi)\right)-1\,,\qquad \rho\neq 0 \quad (21)$$

Define the GE loss function:

$$L_{\text{GE}}\left(l(\varphi),\hat{l}(\varphi)\right)=\left(\frac{\hat{l}(\varphi)}{l(\varphi)}\right)^{\theta}-\theta\ln\left(\frac{\hat{l}(\varphi)}{l(\varphi)}\right)-1\,,\quad \theta\neq 0 \quad (22)$$

The SE loss function produces the BEs of $l(\varphi)$ as follows:

$$\hat{l}_{SEL}(\varphi)=E(l(\varphi)\backslash x) =\int_{\psi} l(\varphi)\,\pi(\varphi\backslash x)d\psi \quad (23)$$

The BEs of $l(\varphi)$ are calculated using the LINEX loss function:

$$\hat{l}_{LINEX}(\varphi)=-\frac{1}{\rho}\ln\left(E_{\varphi}\left[e^{-\rho l(\varphi)}\backslash x\right]\right) \quad (24)$$

and the following is the procedure for calculating the BEs of $l(\varphi)$ using the GE loss function:

$$\hat{l}_{GE}(\varphi)=\left(E_{\varphi}\left[l(\varphi)^{-\theta}\backslash x\right]\right)^{-\frac{1}{\theta}} \quad (25)$$

We used MCMC in R software to estimate unknown parameters, since Equations (23)-(25) are analytically challenging to calculate. Subsequently, we used the MCMC methodology to produce posterior samples and identify appropriate BEs. MCMC is an effective simulation method for predicting significant values and sampling from posterior distributions. Bayesian estimates may be computed via the MCMC method and the three functions. Upon removing burn-in from the random sample size Q derived from the posterior density, Bayes estimates may be computed using the residual samples. The Bayesian estimators of $\varphi^{(i)}=\left(\mathrm{b}^{(i)},\mathrm{a}^{(i)},\mathrm{c}^{(i)}\right)$ were derived using MCMC under the SEL, LINEX, and GEL functions as follows:

$$\hat{\varphi}_{SEL}=\frac{1}{Q-l_B}\sum_{i=l_B}^{Q}\varphi^{(i)} \quad (26)$$

$$\hat{\varphi}_{LINEX}=-\frac{1}{\rho}\ln\left(\frac{1}{Q-l_B}\sum_{i=l_B}^{Q}e^{-\rho\varphi^{(i)}}\right) \quad (27)$$

and

$$\hat{\varphi}_{GE}=\left(\frac{1}{Q-l_B}\sum_{i=l_B}^{Q}\left[\varphi^{(i)}\right]^{-\theta}\right)^{-\frac{1}{\theta}} \quad (28)$$

where l_B is the burn-in duration of the MCMC.

5. Simulation Study

This paper evaluates the BXII-IE distribution par ameters using both Bayesian and non-Bayesian e stimators. The non-Bayesian estimate was condu cted using the same methods as before: MLE, LS E, WLSE, MPSE, and ADE. With n=30, 60, 12 0, 200, and 300, create 1000 BXII-IE random sa mples of size n. sets of parameters were consider ed, with (b=3, a=1, c=0.4) and (b=3.1, a=1.1, c= 0.7), respectively. As shown before, SE, LINEX , and GE were the loss functions used for the Ba yesian estimate. Several possible outcomes with different values for (b=0.4, a=2.2, c=1.4), (b=0.5 , a=2.2, c=1.7) and (b=0.5, a=2, c=1.5) were con sidered in the simulation. They examined two ρ values, -0.7 and 0.7, in the LINEX loss function. In the same way, we considered θ = −0.7 and 0.7 with the GE loss function and $s_1=s_2=s_3=p_1=p_2=p_3=1.5$. Each random sample had a size of 50, 100, and 200. The estimator's perform ance was assessed using results such as bias, root mean squared error (RMSE), Highest Posterior Density Intervals (HPD), average interval length (AIL), and coverage probability (CP) for Bayesi an regions, respectively. Here are the results of t he non-Bayes estimates (Tables 1-2) and the Bay es estimations (Tables 3-5). The tabulated values showed these results. When it comes to classical estimating, ADE and MLE are the most efficient estimators. By employing ρ = θ = -0.7, the BEs f or the GE and LINEX loss functions may be opti mally calculated using Bayes estimations. Those BEs that used the LN-0.7 and GE 0.7 loss functi ons outperformed those that used the SEL, LN0. 7, and GE -0.7 loss functions.

Table 1: Non-Bayesian BXII-IE model mean, RMSE, and bias with parameters b=3, a=1, c=0.4.

n		E. Par.	MLE	LSE	WLSE	MPSE	ADE
30	Mean	$\hat{b}$	3.2083	3.5932	3.3809	3.5031	3.3769
		$\hat{a}$	2.9708	2.5278	2.3065	3.1449	2.1998
		$\hat{c}$	0.7002	0.5124	0.4967	0.8568	0.4949
	RMSE	$\hat{b}$	2.3187	2.7936	2.2087	3.3394	2.1248
		$\hat{a}$	4.8881	3.5876	3.3900	5.4640	3.2652
		$\hat{c}$	0.7079	0.2743	0.2630	0.9135	0.2680
	Bias	$\hat{b}$	0.2083	0.5932	0.3809	0.5031	0.3769
		$\hat{a}$	1.9708	1.5278	1.3065	2.1449	1.1993
		$\hat{c}$	0.3002	0.1124	0.0967	0.4568	0.0949
60	Mean	$\hat{b}$	3.0812	3.2064	3.1359	3.1309	3.1775
		$\hat{a}$	1.9327	1.8673	1.6239	2.1285	1.4602
		$\hat{c}$	0.5429	0.4621	0.4482	0.6421	0.4383
	RMSE	$\hat{b}$	1.2142	1.4197	1.1075	1.5196	1.0508
		$\hat{a}$	3.3799	2.6092	2.2835	3.7977	1.7794
		$\hat{c}$	0.4993	0.2014	0.1828	0.6635	0.1630
	Bias	$\hat{b}$	0.0812	0.2064	0.1359	0.1309	0.1775
		$\hat{a}$	0.9327	0.8673	0.6239	1.1285	0.4602
		$\hat{c}$	0.1429	0.0621	0.0482	0.2421	0.0383
120	Mean	$\hat{b}$	3.0602	3.0829	3.0740	3.0639	3.0940
		$\hat{a}$	1.2384	1.4102	1.1981	1.4049	1.1935
		$\hat{c}$	0.4411	0.4300	0.4165	0.4823	0.0919
	RMSE	$\hat{b}$	0.7206	0.8511	0.7048	0.8323	0.4148
		$\hat{a}$	1.4388	1.6472	1.0291	2.0407	0.6934
		$\hat{c}$	0.2596	0.1279	0.0934	0.3718	1.1698
	Bias	$\hat{b}$	0.0602	0.0829	0.0740	0.0639	0.0940
		$\hat{a}$	0.2384	0.4102	0.1981	0.4049	0.1935
		$\hat{c}$	0.0411	0.0300	0.0165	0.0823	0.0148
200	Mean	$\hat{b}$	3.0293	3.0401	3.0307	3.0400	3.0449
		$\hat{a}$	1.0954	1.2434	1.1398	1.1887	1.1058
		$\hat{c}$	0.4106	0.4186	0.4110	0.4225	0.4085
	RMSE	$\hat{b}$	0.4725	0.6325	0.5152	0.5182	0.4979
		$\hat{a}$	0.5282	1.2093	0.8757	1.0898	0.6996
		$\hat{c}$	0.0809	0.0966	0.0711	0.1525	0.0591
	Bias	$\hat{b}$	0.0293	0.0401	0.0307	0.0400	0.0449
		$\hat{a}$	0.0954	0.2434	0.1398	0.1887	0.1058
		$\hat{c}$	0.0106	0.0186	0.0110	0.0225	0.0085
300	Mean	$\hat{b}$	3.0177	3.0384	3.0249	3.0265	3.0321
		$\hat{a}$	1.0528	1.1017	1.0561	1.0875	1.0495
		$\hat{c}$	0.4049	0.4080	0.4047	0.4083	0.4041
	RMSE	$\hat{b}$	0.3720	0.4960	0.4028	0.3924	0.3977
		$\hat{a}$	0.2820	0.5413	0.3429	0.3380	0.3196
		$\hat{c}$	0.0320	0.0520	0.0368	0.0361	0.0351
	Bias	$\hat{b}$	0.0177	0.0384	0.0249	0.0265	0.0321
		$\hat{a}$	0.0528	0.1017	0.0561	0.0875	0.0495
		$\hat{c}$	0.0049	0.0080	0.0047	0.0083	0.0041

Table 2: Non-Bayesian BXII-IE model mean, RMSE, and bias with parameters b=3.1, a=1.1, c=0.7

N		Est.Par.	MLE	LSE	WLSE	MPSE	ADE
30	Mean	$\hat{b}$	3.3365	3.6769	3.4751	3.5034	3.4757
		$\hat{a}$	3.1720	2.9236	2.6486	3.3318	2.6063
		$\hat{c}$	1.2240	0.8952	0.8627	1.5218	0.8642
	RMSE	$\hat{b}$	2.8388	2.7772	2.2077	3.0479	2.1327
		$\hat{a}$	5.2602	4.2171	3.9449	5.8744	4.0349
		$\hat{c}$	1.2326	0.0473	0.4350	1.6107	0.4565
	Bias	$\hat{b}$	0.2365	0.5769	0.3751	0.4034	0.3757
		$\hat{a}$	2.0720	1.8236	1.5486	2.2318	1.5063
		$\hat{c}$	0.5240	0.1952	0.1627	0.8215	0.1642
60	Mean	$\hat{b}$	3.1553	3.2964	3.2273	3.1784	3.2696
		$\hat{a}$	2.0458	2.2207	1.8572	2.2528	1.7380
		$\hat{c}$	0.9597	0.8125	0.7896	1.1357	0.7761
	RMSE	$\hat{b}$	1.2260	1.4341	1.1267	1.4543	1.0772
		$\hat{a}$	3.4801	3.2719	2.7032	3.9798	2.4153
		$\hat{c}$	0.8802	0.3555	0.3286	1.1497	0.3113
	Bias	$\hat{b}$	0.0553	0.1964	0.1273	0.0784	0.1696
		$\hat{a}$	0.9458	1.1207	0.7572	1.1528	0.6380
		$\hat{c}$	0.2597	0.1125	0.0896	0.4354	0.0761
120	Mean	$\hat{b}$	3.1495	3.1761	3.1699	3.1462	3.1928
		$\hat{a}$	1.3721	1.6184	1.3510	1.46311	1.3445
		$\hat{c}$	0.7751	0.7555	0.7312	0.8512	0.7268
	RMSE	$\hat{b}$	0.7364	0.8709	0.7216	0.8495	0.7078
		$\hat{a}$	1.59255	1.9652	1.2730	1.7263	1.4364
		$\hat{c}$	0.4502	0.2253	0.1697	0.6691	0.1643
	Bias	$\hat{b}$	0.0495	0.0761	0.0699	0.0462	0.0928
		$\hat{a}$	0.2721	0.5184	0.2510	0.3631	0.2445
		$\hat{c}$	0.0751	0.0555	0.0312	0.1512	0.0268
200	Mean	$\hat{b}$	3.1210	3.1366	3.1285	3.1267	3.1443
		$\hat{a}$	1.2373	1.4186	1.2606	1.3054	1.2333
		$\hat{c}$	0.7265	0.7339	0.7204	0.7489	0.7153
	RMSE	$\hat{b}$	0.4911	0.6481	0.5274	0.5394	0.5087
		$\hat{a}$	0.8890	1.5600	0.9366	1.0527	0.8822
		$\hat{c}$	0.2236	0.1687	0.1285	0.3143	0.1034
	Bias	$\hat{b}$	0.0210	0.0366	0.0285	0.0267	0.0443
		$\hat{a}$	0.1373	0.3186	0.1606	0.2054	0.1333
		$\hat{c}$	0.0265	0.0339	0.0204	0.0489	0.0153
300	Mean	$\hat{b}$	3.1132	3.1346	3.1229	3.1187	3.1311
		$\hat{a}$	1.1665	1.2460	1.1716	1.2133	1.1616
		$\hat{c}$	0.7092	0.7164	0.7088	0.7160	0.7075
	RMSE	$\hat{b}$	0.3769	0.5117	0.4116	0.3986	0.4055
		$\hat{a}$	0.3274	0.7948	0.4127	0.4201	0.3750
		$\hat{c}$	0.0560	0.1022	0.0657	0.0653	0.0617
	Bias	$\hat{b}$	0.0132	0.0346	0.02299	0.0187	0.0311
		$\hat{a}$	0.0665	0.1460	0.0716	0.1133	0.0616
		$\hat{c}$	0.0095	0.0164	0.0088	0.0160	0.0075

Table 3: Bayesian BXII-IE model MSE, RMSE, Bias, HPD, AIL, and CP with parameters b=0.4, a=2.2, c=1.4

M.			**50**			**100**			**200**	
	Par.	b	a	c	b	a	c	b	a	c
SEL	MSE	0.2504	0.8053	0.2494	0.1952	0.6283	0.2059	0.1558	0.4893	0.1719
	RMSE	0.5004	0.8974	0.4994	0.4419	0.7926	0.4538	0.3947	0.6995	0.4146
	Bias	0.4210	0.5472	0.3774	0.3734	0.3996	0.3541	0.3323	0.2922	0.3328
	HPD_L	0.3001	0.7339	0.2694	0.3608	0.9099	0.3260	0.3712	0.9276	0.3077
	HPD_U	1.3442	3.4708	1.4186	1.2358	3.3786	1.4057	1.1376	3.3854	1.3055
	AIL	1.0440	2.7369	1.1491	0.8750	2.4686	1.0797	0.7664	2.4578	0.9977
	CP	96	96.80	96.30	96.30	96.30	97.30	95.80	97	97.10
LN-0.7	MSE	0.2760	0.7632	0.2508	0.2136	0.5919	0.2026	0.1665	0.4609	0.1674
	RMSE	0.5254	0.8736	0.5008	0.4622	0.7693	0.4501	0.4081	0.6789	0.4092
	Bias	0.4404	0.4327	0.3531	0.3880	0.3015	0.3354	0.3417	0.2199	0.3197
	HPD_L	0.3014	0.7934	0.2801	0.3423	0.9416	0.3301	0.3710	0.9748	0.3082
	HPD_U	1.4126	3.6637	1.4624	1.2539	3.6184	1.4942	1.2472	3.5210	1.3376
	AIL	1.1112	2.8703	1.1822	0.9115	2.6767	1.1641	0.8761	2.5462	1.0293
	CP	96	96.40	96.20	95.80	97	97.50	96.90	97.50	97.20
LN 0.7	MSE	0.2283	0.8760	0.2522	0.1795	0.6835	0.2106	0.1461	0.5265	0.1768
	RMSE	0.4778	0.9359	0.5022	0.4237	0.8267	0.4589	0.3823	0.7256	0.4204
	Bias	0.4029	0.6476	0.3991	0.3597	0.4891	0.3714	0.3234	0.3603	0.3452
	HPD_L	0.2977	0.6994	0.2665	0.3943	0.8040	0.3265	0.3534	0.7623	0.3145
	HPD_U	1.3197	3.1733	1.4054	1.2397	3.4161	1.2763	1.1247	3.3464	1.2206
	AIL	1.0220	2.4739	1.1389	0.8454	2.6120	0.9498	0.7712	2.5840	0.9061
	CP	96.20	95.80	97	97.20	97.30	96.10	96.20	97.20	96.20
GE-0.7	MSE	0.2414	0.8298	0.2538	0.1886	0.6470	0.2104	0.1516	0.5018	0.1754
	RMSE	0.4913	0.9109	0.5038	0.4342	0.8043	0.4587	0.3894	0.7084	0.4189
	Bias	0.4125	0.5705	0.3880	0.3667	0.4195	0.3629	0.3278	0.3068	0.3393
	HPD_L	0.2969	0.7204	0.2653	0.3488	0.8155	0.3018	0.3744	0.7853	0.3142
	HPD_U	1.3077	3.3945	1.3523	1.2378	3.3846	1.3051	1.2234	3.3764	1.2475
	AIL	1.0108	2.6741	1.0870	0.8889	2.5690	1.0033	0.8490	2.5910	0.9333
	CP	95.50	96.10	95.90	96.60	96.40	95.90	97.40	96.90	96.30
GE 0.7	MSE	0.2026	0.9587	0.2776	0.1600	0.7462	0.2329	0.1333	0.5664	0.1929
	RMSE	0.4501	0.9791	0.5269	0.4000	0.8638	0.4826	0.3652	0.7526	0.4392
	Bias	0.3731	0.6771	0.4351	0.3360	0.5118	0.4022	0.3070	0.3748	0.3685
	HPD_L	0.2876	0.6485	0.2300	0.3342	0.7327	0.2689	0.3679	0.7633	0.3037
	HPD_U	1.3077	3.4429	1.2572	1.1836	3.2477	1.2329	1.1315	3.2861	1.2230
	AIL	1.0201	2.7943	1.0272	0.8494	2.5149	0.9640	0.7635	2.5227	0.91923
	CP	96.90	97.60	96.20	97.20	96	96.20	97	96.30	96.80

Table 4: Bayesian BXII-IE model MSE, RMSE, Bias, HPD, AIL, and CP with parameters b=0.5, a=2.2, c=1.7

M.			**50**			**100**			**200**	
	Par.	b	a	c	b	a	c	b	a	c
SEL	MSE	0.2749	0.5969	0.4519	0.2133	0.4667	0.3742	0.1660	0.3680	0.3015
	RMSE	0.5243	0.7726	0.6722	0.4618	0.6831	0.6117	0.4074	0.6067	0.5491
	Bias	0.4564	0.5159	0.5462	0.3992	0.3698	0.4766	0.3464	0.2511	0.4156
	HPD_L	0.4980	0.6671	0.5398	0.4846	0.8234	0.5598	0.4757	0.8885	0.6122
	HPD_U	1.5210	2.9123	2.0007	1.3238	2.9640	2.9640	1.2686	2.9953	2.0605
	AIL	1.0230	2.2451	1.4609	0.8392	2.1406	1.5070	0.7928	2.1067	1.4482
	CP	97.20	96.50	96.50	96.50	96.50	97	95.60	96.90	97.30
LN-0.7	MSE	0.3060	0.5628	0.4386	0.2354	0.4402	0.3587	0.1783	0.3534	0.2878
	RMSE	0.5532	0.7502	0.6623	0.4852	0.6634	0.5989	0.4222	0.5944	0.5364
	Bias	0.4801	0.4115	0.4915	0.4171	0.2778	0.4296	0.3575	0.1797	0.3790
	HPD_L	0.5065	0.6777	0.5352	0.4648	0.7789	0.5322	0.4583	0.9082	0.6311
	HPD_U	1.5774	3.0155	2.1179	1.3590	3.1373	2.2035	1.3450	3.1397	2.1747
	AIL	1.0709	2.3378	1.5826	0.8942	2.3583	1.6712	0.8866	2.2314	1.5435
	CP	97.20	95.70	0.6623	96.10	96.60	97	96.70	97.40	97.70

LN 0.7	MSE	0.2478	0.6558	0.4776	0.1941	0.5115	0.3961	0.1548	0.3927	0.3186
	RMSE	0.4978	0.8098	0.6911	0.4406	0.7152	0.6294	0.3935	0.6267	0.5644
	Bias	0.4342	0.6067	0.5923	0.3824	0.4524	0.5175	0.3356	0.3169	0.4485
	HPD_L	0.5031	0.6687	0.5326	0.4612	0.6703	0.6156	0.4739	0.7702	0.6104
	HPD_U	1.4775	2.6508	1.9511	1.2727	2.9125	1.9226	1.2664	3.0145	1.9196
	AIL	0.9743	1.9821	1.4185	0.8114	2.2422	1.3070	0.7925	2.2443	1.3091
	CP	97.60	95.70	97.20	95.80	97	96.60	96.20	97.40	96.30
GE-0.7	MSE	0.2647	0.6180	0.4633	0.2058	0.4827	0.3841	0.1615	0.3775	0.3086
	RMSE	0.5145	0.7861	0.6807	0.4537	0.6948	0.6198	0.4019	0.6144	0.5555
	Bias	0.44712	0.5396	0.5618	0.3919	0.3905	0.4902	0.3415	0.2666	0.4262
	HPD_L	0.4984	0.6600	0.5408	0.4614	0.7228	0.5573	0.4741	0.7710	0.6282
	HPD_U	1.4781	2.8122	1.9358	1.3164	2.9521	1.9417	1.3213	2.9862	1.9826
	AIL	0.9797	2.1521	1.3950	0.8549	2.2292	1.3844	0.8471	2.2151	1.3543
	CP	96.70	95.90	96.20	96.50	96.60	95.90	97	97	96.70
GE 0.7	MSE	0.2208	0.7308	0.5229	0.1736	0.5698	0.4337	0.1419	0.4280	0.3438
	RMSE	0.4699	0.8548	0.7231	0.4167	0.7548	0.6586	0.3767	0.6542	0.5863
	Bias	0.4038	0.6470	0.6300	0.3581	0.4851	0.5500	0.3187	0.3379	0.4734
	HPD_L	0.4485	0.5984	0.4921	0.4420	0.6829	0.5445	0.4590	0.7934	0.5865
	HPD_U	1.4283	2.8157	1.8270	1.3038	2.8199	1.8968	1.2536	2.9384	1.9535
	AIL	0.9798	2.2173	1.3349	0.8617	2.1370	1.3523	0.7945	2.1450	1.3670
	CP	97.50	97.20	96.40	97.20	96.10	96.50	96.60	970	97.10

Table 5: Bayesian BXII-IE model MSE, RMSE, Bias, HPD, AIL, and CP with parameters b=0.5, a=2, c=1.5

M.			**50**			**100**			**200**	
	Par.	b	a	c	B	a	c	b	a	c
SEL	MSE	0.2312	0.4479	0.3118	0.1808	0.3602	0.1808	0.1440	0.2954	0.2108
	RMSE	0.4808	0.6693	0.5584	0.4252	0.6002	0.5064	0.3795	0.5435	0.4591
	Bias	0.4112	0.3968	0.4126	0.3604	0.2718	0.3539	0.3165	0.1811	0.3089
	HPD_L	0.4453	0.6168	0.4958	0.4411	0.8329	0.4922	0.4551	0.8938	0.5851
	HPD_U	1.4318	2.7105	1.9068	1.2954	2.8598	1.9278	1.2423	2.9075	1.9963
	AIL	0.9865	2.0937	1.4110	0.8542	2.0269	1.4356	0.7871	2.0136	1.4111
	CP	96.60	96.50	96.60	95.80	96.80	97.10	95.90	98	98
LN-0.7	MSE	0.2569	0.4334	0.3102	0.2007	0.3490	0.2532	0.1554	0.2876	0.2047
	RMSE	0.5069	0.6583	0.5569	0.4480	0.5908	0.5032	0.3942	0.5363	0.4524
	Bias	0.4325	0.3092	0.3652	0.3771	0.1951	0.3127	0.3271	0.1240	0.2790
	HPD_L	0.4580	0.6202	0.4885	0.4426	0.8152	0.5632	0.4509	0.8320	0.5872
	HPD_U	1.4966	2.8110	2.0090	1.3325	2.9851	2.1178	1.3314	2.9205	2.0379
	AIL	1.0385	2.1907	1.5205	0.8898	2.1698	1.5545	0.8805	2.0884	1.4507
	CP	96.70	95.60	96.10	96	97.10	97.70	96.90	97.40	980
LN 0.7	MSE	0.2089	0.4819	0.3242	0.1641	0.3850	0.2664	0.1339	0.3101	0.2200
	RMSE	0.4571	0.6942	0.5694	0.4051	0.6204	0.5162	0.3660	0.5569	0.4690
	Bias	0.3913	0.4743	0.4534	0.3448	0.3417	0.3902	0.3064	0.2348	0.3361
	HPD_L	0.4145	0.6817	0.4397	0.4602	0.7837	0.5908	0.4532	0.7262	0.5338
	HPD_U	1.3635	2.5779	1.8033	1.2802	2.8362	1.8080	1.2338	2.8189	1.7886
	AIL	0.9489	1.8961	1.3635	0.8200	2.0525	1.2172	0.7806	2.0926	1.2548
	CP	96.20	96.80	96.90	96.40	97.80	96.80	96.50	97.50	96
GE-0.7	MSE	0.2224	0.4622	0.3190	0.1741	0.3710	0.2626	0.1399	0.3022	0.2155
	RMSE	0.4716	0.6798	0.5648	0.4173	0.6091	0.5124	0.3741	0.5498	0.4642
	Bias	0.4025	0.4179	0.4272	0.3534	0.2902	0.3666	0.3118	0.1947	0.3183
	HPD_L	0.4312	0.7121	0.4947	0.4396	0.7980	0.5648	0.4451	0.7851	0.5433
	HPD_U	1.3708	2.7231	1.8409	1.3050	2.8597	1.8576	1.2857	2.8934	1.8595
	AIL	0.9396	2.0109	1.3462	0.8653	2.0617	1.2928	0.8405	2.1082	01.316
	CP	95.70	97.30	96.20	96.50	97.20	96.50	97.10	98	96.20
	MSE	0.1852	0.5411	0.3586	0.1459	0.4315	0.2949	0.1220	0.3395	0.2394
	RMSE	0.4304	0.7356	0.5988	0.3820	0.6569	0.5431	0.3494	0.5827	0.4893

GE 0.7	Bias	0.3623	0.51	0.4913	0.3212	0.3750	0.4231	0.2900	0.2576	0.3609
	HPD_L	0.3773	0.5772	0.4479	0.4225	0.7191	0.5392	0.4606	0.7943	0.5264
	HPD_U	1.3355	2.6135	1.7342	1.2646	2.7441	1.8013	1.2457	2.8473	1.8316
	AIL	0.9582	2.0362	1.2862	0.8420	2.0250	1.2620	0.7850	2.0530	1.3052
	CP	96.60	97.30	96.50	97.10	96.60	97.10	97.20	97.90	96.90

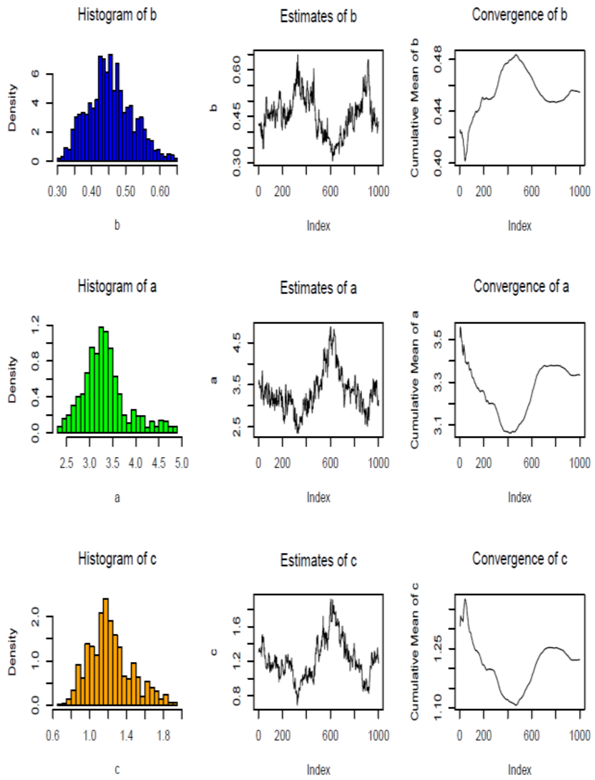

Figure 3. MCMC Plot for the parameter in Table 3 of the BXII-IE when n=200

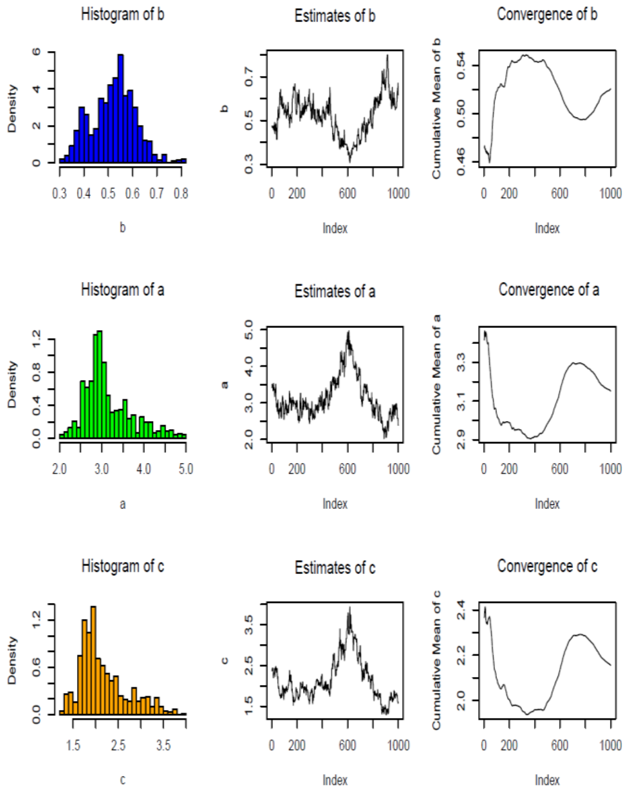

Figure 4. MCMC Plot for the parameter in Table 4 of th e BXII-IE when n=200

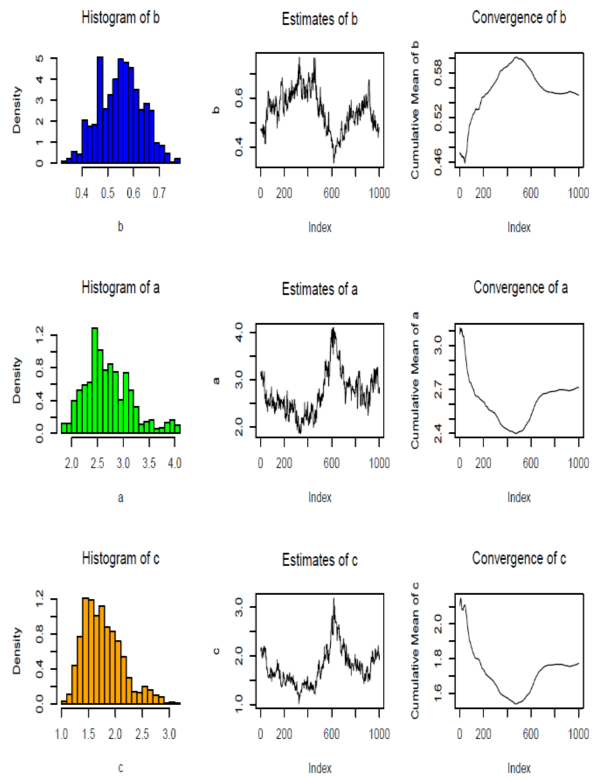

Figure 5. MCMC Plot for the parameter in Table 5 of th e BXII-IE when n=200.

6. Application

This section examines a particular application employing COVID-19 statistics of 36 Canadian COVID-19-related fatalities from April 10 to May 15, 2020 [42]: (3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091,3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048) We want to show BXII-IE distribution fitting. The BXII-IE distribution is applied to data and compared to GRIW [43], TEEIW [44], and OEKIE [45] models for flexibility. We use log-likelihood, Akaike Information Criterion (AIC), corrected AIC, Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Anderson-Darling statistic (A), Cramer-von Mises statistic (W), Kolmogorov-Smirnov, and P-value statistic to ensure a complete comparison. Finding the best data model is our objective.

Table 6: The goodness-of-fit, and information criteria for Canadian COVID-19 data.

Model	MLEs	-LL	AIC	CAIC	BIC	HQIC	W	A	KS	P-V
BXII-IE	$\hat{b}$:2.8256 $\hat{a}$:1.8538 $\hat{c}$:1.7739	47.38	100.77	101.52	105.52	102.43	0.0726	0.4273	0.1071	0.8097
GRIW	$\hat{b}$:0.1186 $\hat{a}$:5.5727 $\hat{c}$:0.3922 $\hat{d}$:0.6788	48.11	104.24	105.52	110.56	106.44	0.0970	0.5678	0.1085	0.8011
TEEIW	$\hat{b}$:2.0640 $\hat{a}$:0.8947 $\hat{c}$:4.2086 $\hat{d}$:1.2029	47.96	103.92	105.21	110.25	106.13	0.0941	0.5526	0.1134	0.7431
OEKIE	$\hat{b}$:4.0427 $\hat{a}$:1.0445 $\hat{c}$:0.2260 $\hat{d}$:3.4710 $\hat{e}$:3.4210	48.62	107.24	109.24	115.16	110.01	0.0969	0.5633	0.1065	0.8084

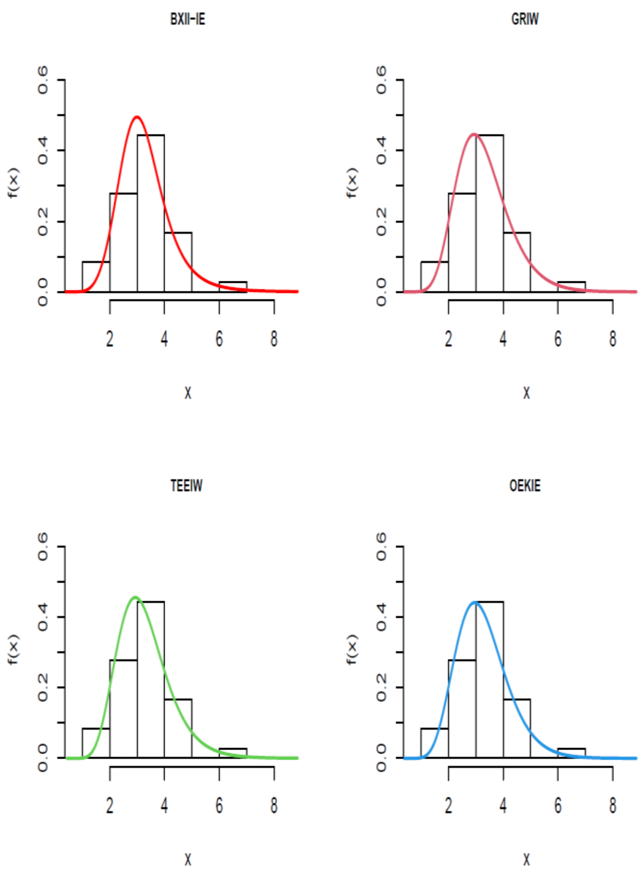

Figure 6. The empirical PDF for Canadian COVID-19 data

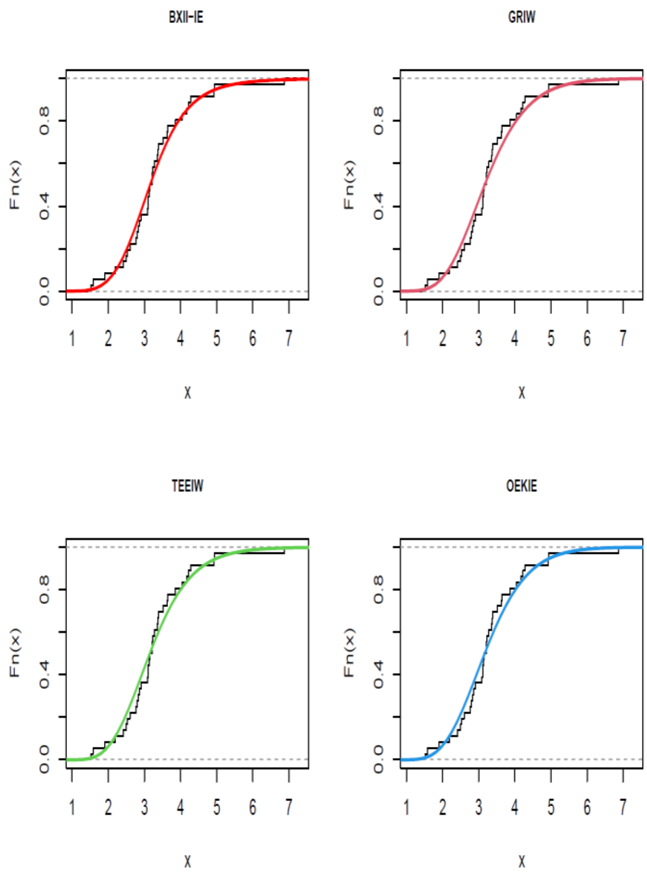

Figure 7. The empirical CDF for Canadian COVID-19 data

Table 6 shows BXII-IE model parameter estimates, log-likelihood estimates, and statistics like AIC, CAIC, BIC, HQIC, W, A, KS, and the highest p-value, out of all models, BXII-IE has the lowest statistics. Thus, the BXII-IE model best fits data and is a competitive model. Figures

6 and 7 show the Empirical PDF, CDF plots of the BXII-IE model, and comparative models.

7. Conclusions

This study aims to estimate the parameters of the BXII-IE distribution using two distinct types of e stimators: classical and Bayesian. We explore bo th symmetric and asymmetric loss functions to e nhance the robustness and accuracy of our estim ates. Specifically, the Bayesian estimators are de rived using three loss functions: squared error (S E), general entropy (GE), and LINEX. To achiev e this, we employ suitable prior distributions for the parameters involved.

Due to the absence of closed-form solutions for t he Bayesian estimates under these specified loss functions, we utilized the MCMC technique to fa cilitate the estimation process. This method allo ws for efficient sampling from posterior distribut ions, enabling us to obtain reliable parameter esti mates.

The application of the proposed BXII-IE model t o COVID-19 mortality rate data from Canada de monstrates its superiority over several competiti ve models

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