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Estimation Parameters in the fuzzy class Poisson mixture regression model

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ABSTRACT

In the Poisson mixture regression for the fuzzy class model, observations originate from distinct sub-sources or classes due to the problem of heteroscedasticity. The underlying assumption posits that the observed data stem from a finite mixture of fuzzy classes. The primary challenge lies in achieving the optimal assignment of observations to their respective categories, necessitating the utilization of sophisticated methods for parameter estimation within the model. This research paper explores the estimation of parameters in a Poisson mixture regression model designed for the fuzzy class. Leveraging the FCML (Fuzzy Classification Maximum Likelihood algorithm) and genetic algorithm methodologies, we conducted simulations to assess the accuracy and efficacy of parameter estimation. Our findings reveal a clear superiority of the genetic algorithm over the FCML algorithm, as evidenced by the Mean Square Error (MSE) criterion. The genetic algorithm consistently produced lower MSE values, indicating more precise and reliable parameter estimates compared to the FCML algorithm. This research underscores the potential of the genetic algorithm as a robust and effective tool for parameter estimation in complex statistical models, offering researchers an alternative approach for tackling challenges in the estimation of parameters within the context of the fuzzy class

1. Introduction

The analysis of heterogeneous count data presents a significant challenge in statistical modeling, demanding advanced methodologies for parameter estimation. Traditional Poisson regression models often struggle to capture the complexities inherent in such data, especially when observations originate from a mixture of Poisson distributions, complicating both component determination and observation assignment. Consequently, sophisticated parameter estimation techniques are crucial for accurate modeling in these scenarios.

Recent research has made strides in addressing these challenges. Yang and Lai [12] introduced the fuzzy class model in 2005, employing fuzzy set theory and fuzzy classification maximum likelihood (FCML) operations to analyze categorical count data effectively. However, the efficacy of the FCML algorithm within the fuzzy class model context remains uncertain. Bermúdez and Karlis [1] explored bivariate Poisson regression models in 2012, incorporating zero-inflated models for car insurance ratemaking, showcasing significant improvements in modeling automobile insurance claims data. Fu et al. [5], in 2018, proposed the generalized Poisson-multinomial mixture model, addressing count data analysis challenges in grouped and right-censored categories.

Moreover, Gómez and Calderín [6] proposed a mixed Poisson regression model in 2016, integrating the Poisson distribution with the exponential-inverse Gaussian distribution to

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analyze healthcare utilization data among individuals aged 65 and over, effectively addressing over-dispersion and non-user proportion issues. Mufudza and Erol [9], in the same year, utilized Poisson mixture regression models to enhance the prediction and diagnosis of early heart disease, demonstrating superior predictive accuracy and model performance compared to standard Poisson regression models.

In this study, we focus on parameter estimation within the fuzzy class Poisson mixture regression model. We aim to compare the FCML algorithm with an alternative approach, the genetic algorithm, to evaluate their accuracy and efficiency in parameter estimation. Through comprehensive simulations and analysis using the Mean Square Error (MSE) criterion, we seek to discern the superiority of one method over the other in this context.

2. Methodology

2.1 Fuzzy Set

The concept of the fuzzy group originated by Zadeh [14] , who defined the fuzzy group $(\tilde{A})$ to the universal group of elements (X), it is a group of elements with a degree of belonging to each element $\left(M_{\tilde{A}}(x)\right)$ and its value is confined between [0 ,1] , which represents the degree of belonging (x) to the group $(\tilde{A})$. He defined the fuzzy group as classes of elements with a continuous degree of belonging and that this group is characterized by an affiliation function or (Membership Function) that assigned to each element a degree of belonging ranging from zero to one [12] .

2.2 Membership Functions

Membership functions are the most important in fuzzy group theory, as they are used to define the membership of any element of the sample space to the fuzzy set and produce values within the period [0,1] for each element of the overall group to express the degree of belonging of these elements to The fuzzy subgroup. When the element belongs completely to the fuzzy subgroup, then its membership function is one, and when it does not completely belong to the fuzzy subgroup, the value of its affiliation function is equal to zero, and if the value of its membership function is between (0,1), that is, it belongs to the fuzzy subgroup to a certain extent. And if the membership functions contain only two values 0 and 1, then the fuzzy set turns into a Crisp Set, that is, the Fuzzy set is a special case of the fuzzy set [7] .

2.3 Fuzzy c-partition Membership Functions

A type of membership function used in fuzzy clustering algorithms, specifically in the FCML algorithm, the fuzzy c-partition assigns a degree of belonging (a value between 0 and 1) to each data point i with respect to each class k, which indicates the degree to which the data point belongs to that class [2]. The fuzzy c-partition function calculates membership values by measuring the similarity between the data point and class centers [3].
The fuzzy c-partition is usually represented as:

$$\mu_{ki} = \left(\sum_{s=1}^{c} \frac{(d_{ki})^{\frac{1}{m-1}}}{(d_{si})^{\frac{1}{m-1}}} \right)^{-1} \qquad (1)$$

μ_{ki} : is the membership value of data point i in class k
c : Number of class
d_{ki} : The Euclidean distance between data point i and class center k.
d_{si} : is the Euclidean distance between data point i and center of class s.
m : It is a constant that controls the degree of fuzzy .

2.4. Fuzzy Class Poisson Regression

The dependent variable we will deal with is y_i which represents the total number of events measured in a sample containing (n) observations . It is assumed that the independent observations $y_{1\dots}y_n$ consist of (c) classes, and each y_i observation consisting of class k has a Poisson distribution with parameter $\varphi_{i|k}$ as follows [13]:

$$P_k(y_i|\varphi_{i|k}) = \frac{e^{-\varphi_{i|k}}(\varphi_{i|k})^{y_i}}{y_i!} \; . \; i = 1. \dots . n \, . \, k = 1. \dots . c$$

According to the generalized linear model [10] , the traditional parameter is as follows :

$Q(\varphi_{i|k}) = \ln \varphi_{i|k}$

The canonical link function used for Poisson regression models (also known as log-linear models) with explanatory variables $x_{i1} \dots x_{iL}$ is as follows:

$\ln \varphi_{i|k} = \beta_{0k} + x_{i1}\beta_{1k} + \cdots + x_{iL}\beta_{Lk}$ (2)

Typically, the fuzzy class model is used to analyze aggregated discrete data that are assumed to follow a mixture distribution.

Thus, the Poisson regression model for the fuzzy class is a mixture of a linear logarithmic (Poisson) regression distribution with a variable k (k = 1, 2, ..., c)) [13] as follows:

$$P(y_i|\alpha.\beta) = \sum_{k=1}^{c} \alpha_k \, P_k(y_i|\varphi_{i|k}) \qquad (3)$$

Where $\beta = \{\beta_1. \dots . \beta_c\}$ with $\beta_k = \{\beta_{0k}. \beta_{1k}. \dots . \beta_{Lk}\}$

β_{lk} : effect of the explanatory variable l on the mean event rate in class k.

β_{0k} : The average base event rate in class k

α_k : denotes the proportion of class k wher $\sum_{k=1}^{c} \alpha_k = 1$

In Poisson regression for the fuzzy class we will use the fuzzy class variable μ_{ki} , the function is called fuzzy c-partition It is a fuzzy function that takes values in the interval [0.1] , so that it is $\sum_{k=1}^{c} \mu_{ki}(y_i) = 1$ for each y_i .

A model based on these fuzzy class variables can be called a fuzzy class model [13] .

The objective function of the fuzzy class Poisson regression model will be as follows:

$$R_{m,w}(\mu, \alpha, \varphi) = \sum_{k=1}^{c}\sum_{i=1}^{n} \mu_k^m(y_i) \ln p_k(y_i|\varphi_k) + w \sum_{k=1}^{c}\sum_{i=1}^{n} \mu_k^m(y_i) \ln \alpha_k \qquad (4)$$

2.5 Estimation of Poisson mixture regression parameters for fuzzy class using FCML algorithm

Now, we begin to estimate these parameters by maximizing the function $R_{m,w}(\mu, \alpha, \varphi)$, Similarly, with the constraints $\sum_{k=1}^{c} \alpha_k = 1$ and $\sum_{k=1}^{c} \mu_{ki}(y_i) = 1$ we need to consider the Lagrange multiplier

$$R_{m,w}(\mu, \alpha, \varphi) - \lambda_1 \left(\sum_{k=1}^{c} \mu_k - 1 \right) - \lambda_2 \left(\sum_{k=1}^{c} \mu_{ki} - 1 \right)$$

By calculating the derivative with respect to α_k and μ_{ki} respectively, we obtain:

$$\hat{\alpha}_k = \frac{\sum_{i=1}^{n} \mu_{ki}^m}{\sum_{i=1}^{n}\sum_{s=1}^{c} \mu_{si}^m} \qquad (5)$$

$$\hat{\mu}_{ki} = \left(\sum_{s=1}^{c} \frac{(\ln p_{i|k}(y_i|\varphi_{i|k}) + w \ln \hat{\alpha}_k)^{\frac{1}{m-1}}}{(\ln p_{i|s}(y_i|\varphi_{i|s}) + w \ln \hat{\alpha}_s)^{\frac{1}{m-1}}} \right)^{-1} \qquad (6)$$

$i = 1, \dots, n$, $k = 1, \dots . c$

Subsequently, the derivative of $R_{m,w}(\mu, \alpha, \beta)$ with respect to β is computed as follows:

$$\frac{\partial}{\partial \beta_{lk}} R_{m,w}(\mu, \alpha, \varphi) = \sum_{i=1}^{n} \mu_{ik}^m \frac{\partial \ln p_{i|k}(y_i|\varphi_{i|k})}{\partial \varphi_{i|k}} \frac{\partial \varphi_{i|k}}{\partial \beta_{lk}}$$

Where :

$$\frac{\partial \ln p_{i|k}(y_i|\varphi_{i|k})}{\partial \varphi_{i|k}} = \left(\frac{y_i}{\varphi_{i|k}} - 1 \right) \quad , \quad \frac{\partial \varphi_{i|k}}{\partial \beta_{lk}} = x_{il}\varphi_{i|k}$$

$$\frac{\partial}{\partial \beta_{lk}} R_{m,w}(\mu, \alpha, \varphi) = \sum_{i=1}^{n} \mu_{ki}^m (y_i - \varphi_{i|k}) x_{il}$$

The Fisher information matrix is a diagonal matrix :

$$-E\left(\frac{\partial^2 R_{m,w}(\mu, \alpha, \varphi)}{\partial \beta_{lk} \partial \beta_{l'k}} \right) = \sum_{i=1}^{n} \mu_{ki}^m \varphi_{i|k} X_{il} X_{il'} = \{X' W_k X\}_{ll'}$$

$$W_k = diag\{\mu_{ki}^m \varphi_{i|k}\} \qquad (7)$$

We use the Newton-Raphson method, let it be :

$$\gamma_{i|k} = \ln \varphi_{i|k} + (y_i - \varphi_{i|k}) \frac{d \ln \varphi_{i|k}}{d \, \varphi_{i|k}}$$

$$= \ln \varphi_{i|k} + (y_i - \varphi_{i|k}) \frac{1}{\varphi_{i|k}}$$

So :

$X' W_k X \hat{\beta}_k = X' W_k \gamma_k$

$\hat{\beta}_k = (X' W_k X)^{-1} X' W_k \gamma_k$ (8)

Where :$\gamma_k = (\gamma_{1|k}. \dots . \gamma_{n|k})'$

2.6. Genetic algorithm

The genetic algorithm stands as a pivotal instrument within the domain of artificial intelligence and machine learning, drawing inspiration from the principles of biological evolution. This algorithm finds widespread utility across diverse fields, demonstrating its significance in numerous applications [11] . Notably, one of its crucial applications involves enhancing the parameters of regression models. Specifically, within the realm of statistical modeling, the genetic algorithm proves beneficial for refining the parameter values associated with the Fuzzy Class Mixture Poisson regression model. The graphical representation Figure 2 elucidates the procedural framework of the genetic algorithm [8].

2.7 Implementing the genetic algorithm for parameter estimation in the context of the Poisson mixture regression model tailored for fuzzy classes.

We execute the procedural steps of the genetic algorithm on the objective function equation associated with a Poisson mixture regression model for Fuzzy classes, with the aim of estimating the model parameters.

1- Initialization Phase: Commence with the formation of the chromosome, wherein parameter values constitute the genes. Each gene encapsulates a specific parameter.

2- Initialization Step: Generate the inaugural generation by assigning initial values to the genes. These values are generated randomly to establish the starting point for the optimization process.

3- Within the objective function, an assessment of the chromosome takes place, and the one exhibiting a smaller objective function value, indicative of a higher likelihood, is chosen. Subsequently, the evaluation function is determined using the following equation:

$$fitness\ function = \frac{1}{1 + objective\ function}$$

The probability of the evaluation function, specifically the best evaluation, can be computed utilizing the following formula:

$$C_i = \frac{f_{(i)}}{\sum_{i=1}^{N} f_{(i)}}$$

Where :

C_i : denotes the probability of chromosome i

f_i : signifies the evaluation function for chromosome i

N : represents the population size.

Furthermore, employing a selection criterion referred to as the "roulette wheel," a random number denoted as $r_{(c)}$ is generated, falling within the interval [0,1]. Subsequently, this generated number is compared with the value associated with the first chromosome, denoted as $c_{(1)}$ if $r_{(c)}$ is less than $c_{(1)}$, the first chromosome is selected. This iterative process is repeated, determining one chromosome for the new population based on the evaluation function in each iteration.

4- The chosen chromosomes undergo hybridization through mating, utilizing a hybridization criterion referred to as regulated hybridization. This process hinges on the hybridization probability, denoted as P_c a value determined by the researcher, commonly within the range $P_c \geq 0 \cdot 25$. The comparison of this probability value with the genetic values of the chromosomes (parents) is pivotal in generating the new generation (offspring). Specifically, the exchange transpires when the gene value is greater than or equal to the specified P_c .

5- The mutation process is contingent upon the probability value P_m for the parameters. This probability value is calculated utilizing the following formula:

$$P_m = \begin{cases} 0 \cdot 09 - \dfrac{Fitvalue - f_{mean}}{f_{max}} & if\ \ Fitvalue > f_{mean} \\ 0 \cdot 09 & otherwise \end{cases}$$

Where: Fitvalue denotes the evaluation function value , f_{mean} represents the population mean , f_{max}
signifies the maximum value in the population. By substituting the values of randomly selected genes with new values, which are also generated randomly, we obtain these new values using the following formula:

Sum of genes = (number of genes in the chromosome) × (population size).

6- We revisit step three iteratively until the specified convergence criterion is achieved.
7- The parameter evaluation is conducted by leveraging the values of the objective function to estimate the parameters of a Poisson mixture regression model tailored for fuzzy classes.

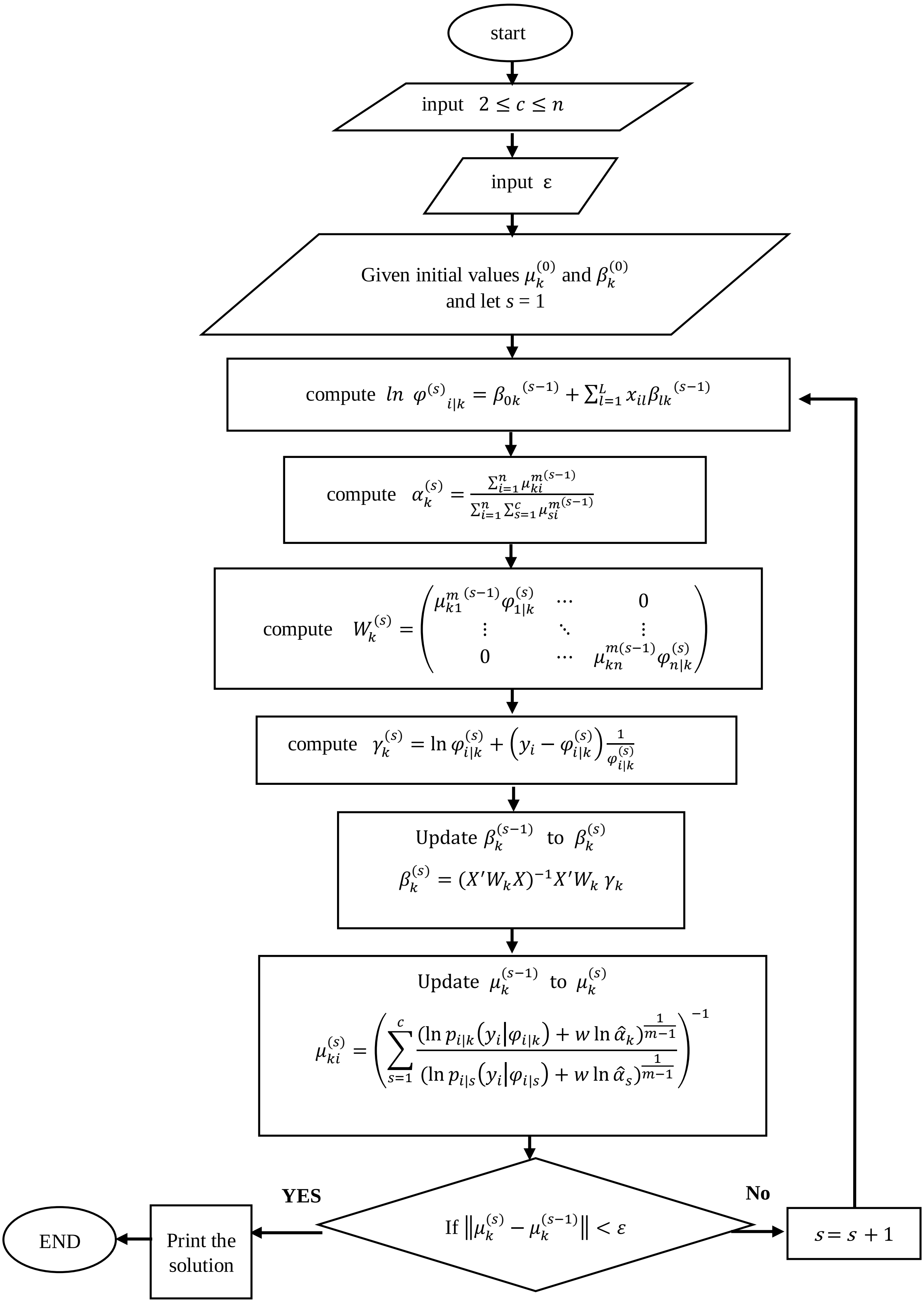

Figure 1:- Diagram of FCML algorithm

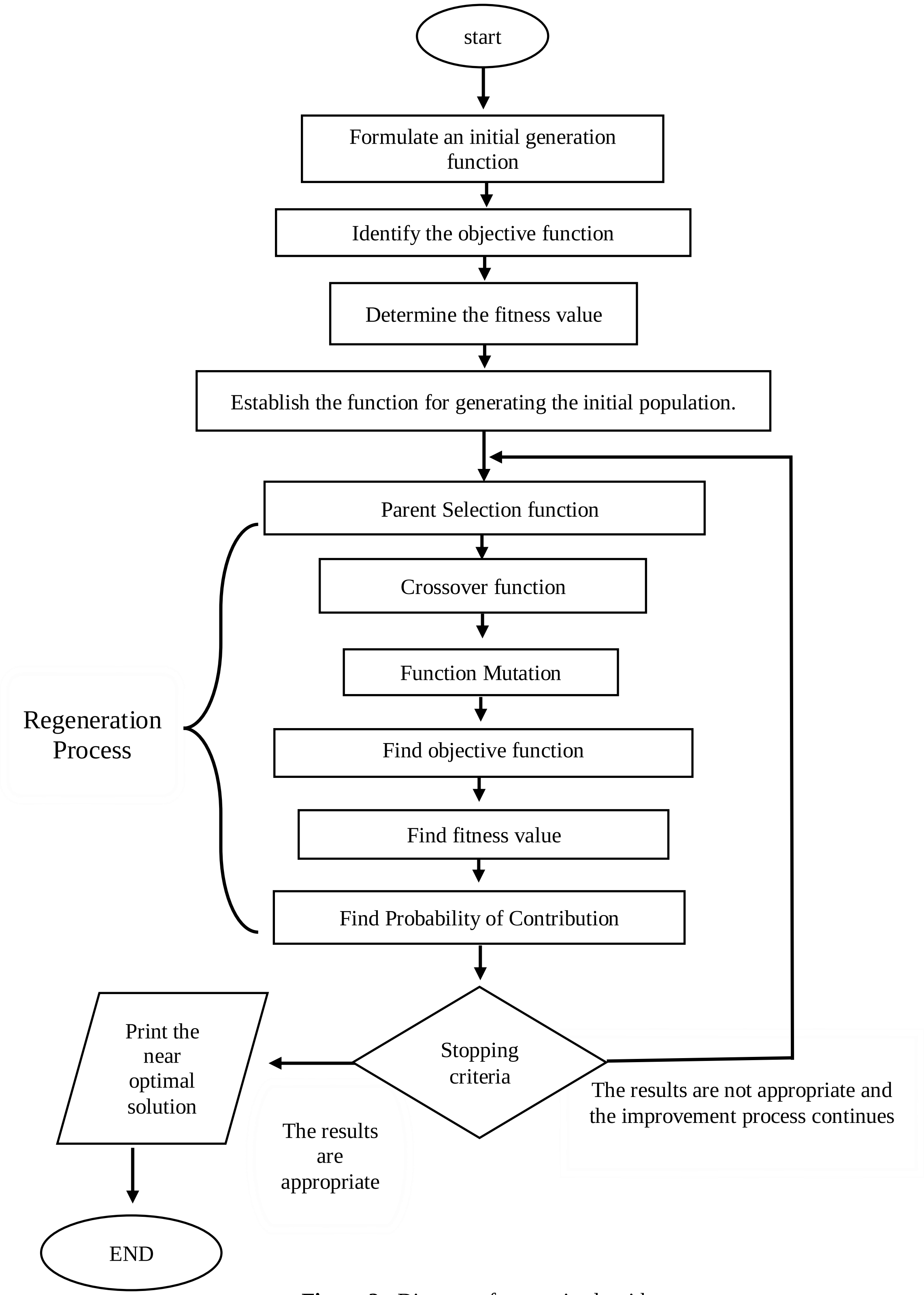

Figure 2: Diagram of a genetic algorithm

3. Discussion of Results

3.1 Simulation Preparation

In this paper, the simulation experiments encompassed the development of multiple programs in R for generating simulated data. This facilitated the comparative analysis of methods across various sample sizes. Three distinct sample sizes, specifically $(n_1 = 50\,, n_2 = 100\,, n_3 = 130)$, were employed for data generation. The independent variables were generated from a uniform distribution:

$x_1 \sim U(0,2) \quad , \quad x_2 \sim U(0,32)$

The model will include one independent variable x_1 so we will use:

$\ln \varphi_{i|k} = \beta_{0k} + \beta_{1k} x_{i1}$

The generation values are based on four models for default parameter values:

Table 1: Represents the three models , M1,M2,M3 and M4, the default parameter values .

Models	β_{01}	β_{11}	β_{02}	β_{12}
M1	1.06	0.92	0.53	-0.72
M2	1.06	0.92	0.02	-1.47
M3	0.51	0.17	0.53	-0.72
M4	0.51	0.17	0.02	-1.47

The parameter alpha (α) is chosen to be 0.3 of the sample size (n) . Accordingly, data will be generated for the first class, amounting to 0.3n.

$\ln \varphi_{i|1} = \beta_{01} + \beta_{11} x_{i1} \qquad i = 1,2,3 \ldots (0.3\, n\,)$

Concerning the observations of the second class, they are generated from the remaining 0.7n based on the model:

$\ln \varphi_{i|2} = \beta_{02} + \beta_{12} x_{i1} \qquad i = (0.3\, n + 1\,) \ldots\; n$

We compute a parameter of the Poisson distribution using the following equation:

$\varphi_{i|k} = e^{\beta_{0k} + \beta_{1k} x_{i1}}$

Subsequently, y_i s generated from a Poisson distribution.

$y_i \sim poisson(\varphi_{i|k})$

Following data generation, estimation methods, specifically the Fuzzy Classification Maximum Likelihood (FCML) algorithm and the Genetic Algorithm (GA), are applied to the dataset.

3.2 Results of Simulation

Each experiment was repeated 500 times. The estimated parameter $\hat{\mu}_{ki}$ is used as a metric for determining the membership of observation's .The calculation involves determining the number of observations correctly classified according to the Rate equation. The Mean Squared Error (MSE) is employed as a statistical measure for the comparative analysis between the estimation methods, namely the Fuzzy Classification Maximum Likelihood (FCML) algorithm and the Genetic Algorithm (GA).

Table 2. Mean Squared Error (MSE) values for the four parameters and models (M1, M2, M3,M4) in relation to the sample size n=50 , $\alpha = 0.3$

Models	Method	$\widehat{\alpha}$	$\hat{\beta}_{01}$	$\hat{\beta}_{11}$	$\hat{\beta}_{02}$	$\hat{\beta}_{12}$	Rate
M1	FCML	0.0045024	0.0836390	0.0452808	0.6126061	0.3574665	0.921
	GA	0.0036157	0.0764711	0.0389424	0.5917904	0.3160768	0.925
M2	FCML	0.0026886	0.0943847	0.0544665	0.4392191	0.7578355	0.956
	GA	0.0025896	0.0670725	0.0407884	0.3335869	0.7000107	0.967
M3	FCML	0.0٠76024	0.1342065	0.3830519	0.2444582	0.2412983	0.759
	GA	0.0023485	0.1233002	0.3412459	0.2054494	0.1637862	0.728
M4	FCML	0.0032641	0.1961105	0.0915367	0.3883045	0.2532273	0.832
	GA	0.002406٧	0.1653157	0.0882319	0.3354334	0.2516412	0.872

Table 3. Mean Squared Error (MSE) values for the four parameters and models (M1, M2, M3,M4) in relation to the sample size n=100 , $\alpha = 0.3$

Models	Method	$\widehat{\alpha}$	$\hat{\beta}_{01}$	$\hat{\beta}_{11}$	$\hat{\beta}_{02}$	$\hat{\beta}_{12}$	Rate
M1	FCML	0.0019902	0.0231743	0.0097550	0.1491733	0.0565184	0.929
	GA	0.0012656	0.0193304	0.0013840	0.0989105	0.0496352	0.958
M2	FCML	0.0026507	0.0244575	0.0186393	0.2964693	0.3823223	0.825
	GA	0.0026012	0.0164515	0.0168955	0.2764693	0.3706431	0.714
M3	FCML	0.004776١	0.0385490	0.1045185	0.0873576	0.1056534	0.773
	GA	0.0024177	0.0220067	0.0977561	0.0338634	0.1035568	0.812
M4	FCML	0.0017109	0.0677538	0.0337335	0.3159017	0.1607795	0.855
	GA	0.0010591	0.0640077	0.0314078	0.3130436	0.1035028	0.864

Table 4. Mean Squared Error (MSE) values for the four parameters and models (M1, M2, M3,M4) in relation to the sample size n=130 , $\alpha = 0.3$

Models	Method	$\widehat{\alpha}$	$\hat{\beta}_{01}$	$\hat{\beta}_{11}$	$\hat{\beta}_{02}$	$\hat{\beta}_{12}$	Rate
M1	FCML	0.0015702	0.0218503	0.0126325	0.1034359	0.0513398	0.926
	GA	0.0008689	0.0159495	0.0030449	0.0982203	0.0350012	0.941
M2	FCML	0.0025455	0.0222975	0.0090846	0.0817681	0.1559543	0.959
	GA	0.0025396	0.0135216	0.0073031	0.0682157	0.1551385	0.963
M3	FCML	0.0004050	0.0065935	0.0486178	0.0486173	0.0482362	0.828
	GA	0.0000778	0.0027772	0.0421718	0.0421718	0.0162375	0.833
M4	FCML	0.0014069	0.0794465	0.0427134	0.1176838	0.052467	0.829
	GA	0.0012654	0.0656766	0.0385020	0.0869124	0.049412	0.833

Upon examination of Table 1, Table 2 and Table 3 it becomes apparent that for a sample size of n=50,100,130 and across all four Models (M1, M2, M3, M4), the Genetic Algorithm outperformed the FCML algorithm. This is evident from the Mean Squared Error (MSE) values for each parameter. Notably, in the Genetic Algorithm, the MSE values for all parameters are consistently lower than those in the FCML algorithm. This suggests that the Genetic Algorithm has effectively enhanced parameter estimation.

4.Conclusion

The research focused on estimating the parameters of a Poisson mixture regression model for the fuzzy class using the FCML algorithm and genetic algorithm. Through extensive simulations, it became evident that the genetic algorithm outperformed the FCML algorithm in terms of parameter estimation. The Mean Square Error (MSE) criterion, employed to assess the accuracy of the estimates, consistently favoured the genetic algorithm, demonstrating its superior performance. The lower MSE values obtained with the genetic algorithm suggest that it provides more precise and reliable parameter estimates for the Poisson mixture regression model in the context of the fuzzy class. This finding underscores the effectiveness of the genetic algorithm as a powerful tool for parameter estimation in complex statistical models, offering researchers a valuable alternative to traditional methods such as the FCML algorithm.

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