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Bayesian Prediction of a Hybrid Spatial Model When It Follows a Multivariate Cauchy Distribution

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ABSTRACT

Spatial models are important economic models that deal with the spatial dimensions of the data of the phenomenon under study. These models have several forms, including the spatial autoregressive model, spatial error, spatial Durbin, and others. In this research, two spatial models were hybridized, namely the spatial autoregressive model and spatial error, and the parameters of the hybrid spatial model were estimated using the Bayesian method when the initial distribution of the parameter to be estimated belongs to the family of known probability distributions when the error of the hybrid spatial model follows a multivariate Cauchy distribution, in addition to finding the predictive distribution of the hybrid model. The researchers concluded that the predictive distribution of the vector of future observations of the hybrid spatial model is an uncommon but appropriate probability distribution (Proper). Through the properties of mathematical expectation, the Bayesian prediction was found. The application was applied to real data related to the number of people with bronchial asthma in Baghdad Governorate for the year (2004). If the number of people with bronchial asthma was studied as a response variable and the variables of lead, carbon monoxide, sulfur dioxide and total suspended particles as variables affecting bronchial asthma, the researchers concluded, based on the (MatlabR2022a) program, that the estimated hybrid spatial model outperformed the estimated general regression model and that the Bayesian method was suitable for conducting the process of predicting future observations, in addition to the fact that the variables of carbon monoxide, sulfur dioxide and total suspended particles have significant effects on the incidence of bronchial asthma

1. Introduction

One of the most important assumptions for studying regression models is the independence of the model error, meaning the absence of a relationship between the errors. However, the existence of a correlation between the observations under study, which is known as autocorrelation, usually violates the basic conditions for completing the model's assumptions. Therefore, the failure to meet the conditions must be considered in the analysis process, as is the case in the analysis of time series data. There are several models related to this, namely the spatial autoregressive model, which stipulates the inclusion of the spatially lagged dependent variable as one of the explanatory variables through the spatial effect parameter that describes the strength of spatial dependence. The Durbin spatial model is also considered one of the spatial autoregressive models, in which the spatial dependence is not only in the dependent variable but also extends to the explanatory variables, while the spatial error model stipulates the existence of a spatial correlation between the error observations and is applicable when spatial autocorrelation

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occurs as a problem resulting from incorrect or insufficient specification of spatial observations. This model aims to correct the spatial error if it exists.

Studied [3] spatial regression model and spatial error model to demonstrate the importance of spatial analysis if the study was conducted on the data of the social and economic household survey for the year (2012). Through this data, it became clear that there is spatial dependence through the Moran test and that ignoring this dependence leads to the loss of important information in the analysis process, in addition to the use of multiple weight matrices that take into account the effect of neighbourhoods between spaces and sites. They concluded that the spatial regression model is superior to the regular regression model by several comparative criteria.

While [12] used the Bayesian method to estimate the parameters of the Durbin Spatial Model (DSM) when the model contains a threshold limit and when the error follows the multi-variate normal distribution. The estimation process for the (DSM) was carried out based on the development of the (Markov Chain Monte Carlo (MCMC)) algorithm. Through the simulation study and based on the comparison criteria, the researcher concluded that when the sample size increases, the efficiency of the estimators increases.

Also studied [4] the effect of falling dust over the Safwan district and the city centre of Basra Governorate and concluded that the falling dust consists of several minerals, including quartz, gypsum and kaolin, and that these particles have mechanical effects on respiratory diseases, in addition to eye, skin and ear infections. [2] reached an assessment of total suspended particles and heavy elements in the district centres of Wasit Governorate, as the concentrations of total particles and some heavy elements such as (Pb, Cr, Fe, Zn, Cd, Cu, Ni) were studied, adding secondary pollutants (NO3, SO4) via an air extraction device during January and July of the year (2019), and the results showed an increase in the concentrations of the studied pollutants exceeding environmental determinants during the summer.

The research dealt with a general introduction to spatial models in the first section, and in the second section, it included a description of the probability function of the multivariate Cauchy distribution. In the third section, the most important spatial models were presented, and the fourth section included the hybridization of some spatial models, represented by the spatial autoregressive model and the spatial error model. The fifth section dealt with the presentation of the standard class adjacency matrix (Weights), and the sixth section included the traditional estimation of the function of the hybrid spatial model and the use of the Lagrange multiplier for the spatial autoregressive model and the spatial error in the seventh section. As for the eighth section, it included the Bayes methodology for estimating the hybrid spatial model, in addition to finding the prediction distribution of the hybrid model in the ninth section. The tenth section dealt with a real applied study of data related to the numbers of people with bronchial asthma in Baghdad Governorate in the year (2004), reviewed the most important theoretical and applied conclusions in the eleventh section.

2. Multivariate Cauchy Distribution:

Cauchy distribution is a continuous probability distribution and a special case of the multivariate (t) distribution, meaning that the multivariate (t) distribution turns into a multivariate Cauchy distribution when the degree of freedom is equal to one. Accordingly, the multivariate Cauchy distribution can be considered a mixed probability distribution, as it results from mixing the multivariate normal distribution and the inverse gamma distribution with parameters $(\frac{1}{2},\frac{1}{2})$ as follows: [6][13]

$$\gamma = inverse\ gamma(\frac{1}{2},\frac{1}{2})$$

The probability function for the inverse gamma distribution is: [6]

$$f(\gamma) = \frac{\gamma^{-\frac{1}{2}-1}}{\sqrt{2\pi}} \quad e^{-\frac{1}{2\gamma}} \quad , \qquad \gamma > 0 \qquad (1)$$

If the description of the random variable $(\underline{Y})$ conditional on the random variable (γ) has a multivariate normal distribution and is defined as follows:

$$(\underline{Y}|\gamma) \sim N_n\left(\underline{\mu}, \gamma\,\sigma^2\, I_n\right)$$

The Probability function for $(\underline{Y}|\gamma)$ is in the following form:

$$f(\underline{Y}|\gamma) = \frac{1}{(2\,\pi\,\gamma\,\sigma^2)^{\frac{n}{2}}}\, e^{-\frac{1}{2\gamma\,\sigma^2}\left(\underline{Y}-\underline{\mu}\right)^T\left(\underline{Y}-\underline{\mu}\right)}\ ,\quad -\infty < \underline{Y} < \infty \qquad (2)$$

By combining equations (1) and (2) and using the concept of Bayes' theorem, the probability function of the multivariate Cauchy distribution is as follows: [10] [7]

$$f(\underline{Y}) = \int_0^{\infty} f(\underline{Y}|\gamma)\, f(\gamma)\, d\gamma$$

$$f(\underline{Y}) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{(\sigma^2)^{\frac{n}{2}}(\pi)^{\frac{n+1}{2}}}\left[1 + \frac{1}{\sigma^2}\left(\underline{Y}-\underline{\mu}\right)^T\left(\underline{Y}-\underline{\mu}\right)\right]^{-\left(\frac{n+1}{2}\right)} \quad (3)$$

Where $(\underline{Y})$ is a random vector of dimension $(n \times 1)$ and $(\underline{\mu})$ is a position parameter vector of dimension $(n \times 1)$ and (σ^2) is the scale parameter of the distribution. Vector $(\underline{Y})$ follows the multivariate Cauchy distribution can be described as follows:

$$\underline{Y} \sim MC_n\left(\underline{\mu}, \sigma^2 I_n\right)$$

3. Spatial Models:

Spatial models are models that are related to phenomena that are spatially close to each other and have similar values of the studied property, meaning that they depend on the concept of the nearest neighbour, i.e. based on the distance between observations. Spatial regression models are very similar to autoregressive models in time series, except that the (lag) will be in the effect of space and not time. There are several other spatial models, including: [3][1]

3.1 Spatial Autoregressive Model (SAR):

(SAR) is a mixed spatial model, where the response variable is included as a spatially lagged variable as one of the variables affecting the model (WY), and (λ) is the parameter that describes the strength of spatial dependence on the response variable. Accordingly, the formula for the spatial autoregressive model (SAR) is shown as follows: [3][1]

$$\underline{Y} = \lambda\, W\underline{Y} + X\underline{\beta} + \underline{\varepsilon} \qquad \ldots (4)$$

Where (W) represents the spatial adjacencies (weights) matrix with a capacity of $(n \times n)$ and (λ) represents the spatial effect parameter that describes the strength of the spatial response and the value of the parameter (λ) is limited between (1) and (-1) and (X) is matrix of influential observations with a dimension of $(n \times k + 1)$ and (k) represents number of influential variables and $(\underline{\beta})$ is vector of model parameters with a capacity of $(k + 1 \times 1)$ and $(\underline{\varepsilon})$ is random error vector with a dimension of $(n \times 1)$.

3.2 Spatial Durbin Model (SDM):

(SDM) is considered one of the spatial models and in the model, there is spatial dependence in the influencing variables, i.e. the matrix of influencing variables is in the form (X.W.X) in addition to the presence of spatial dependence in the response variable. The equation below represents the mathematical formula for the Durbin spatial model: [12] [5] [11]

$$\underline{Y} = \lambda\, W\underline{Y} + [X \quad WX]\,\underline{\beta} + \underline{\varepsilon} \qquad \ldots (5)$$

3.3 Spatial Error Model (SEM):

This model assumes that errors are spatially correlated, unlike the error independence assumptions of the classical model. This model aims to correct for spatial error and the estimators of the ordinary least squares method of this model are unbiased but inefficient at the same time. The mathematical formula of the model is as follows: [3][1]

$$\underline{Y} = X\underline{\beta} + \underline{\varepsilon} \qquad \ldots (6)$$

Whereas:

$$\underline{\varepsilon} = \theta W\underline{\varepsilon} + \underline{u}$$

$$\underline{\varepsilon} = (I - \theta W)^{-1}\underline{u} \quad , \quad \underline{u} \sim N(0, \sigma^2 I)$$

Equation (6) can be rewritten as follows:

$$\underline{Y} = X\underline{\beta} + (I - \theta W)^{-1}\underline{u} \qquad \ldots (7)$$

Since (θ) represents the spatial autoregressive parameter which is estimated with the regression parameters or is the spatial lag coefficient on the error $(\underline{\varepsilon})$, if ($\theta$=0) this means that there is no spatial correlation between the errors for nearby observations, but if ($\theta \neq 0$) this means that there is spatial dependence between nearby observations.

4. Hybrid Spatial Model (HSM):

In this section, two spatial models will be hybridized, namely the spatial autoregressive model and the spatial error model, i.e. the formula in equation (4) will be hybridized with the formula in equation (7), and the mathematical formula for the hybrid spatial model will be as follows:

$$\underline{Y} = \lambda W\underline{Y} + X\underline{\beta} + (I - \theta W)^{-1}\underline{u} \quad \dots(8)$$

$$\underline{Y} = \lambda W\underline{Y} + X\underline{\beta} + \underline{\epsilon} \quad \dots(9)$$

$$Whereas:\ \underline{\epsilon} = (I - \theta W)^{-1}\underline{u}$$

Assuming that the random error of the hybrid model follows a multivariate Cauchy distribution as follows:

$$(\underline{\epsilon}|\gamma) \sim N(0, \gamma\,\sigma^2 (I - \theta W)^{-1})$$

The probability function of the random error vector conditional on the random variable (γ) is according to the following formula:

$$f(\underline{\epsilon}|\gamma) = \frac{1}{(2\,\pi\,\gamma\,\sigma^2)^{\frac{n}{2}}}\, e^{-\frac{1}{2\,\gamma\,\sigma^2}\,\underline{\epsilon}'\underline{\epsilon}} \quad ; \quad -\infty < \underline{\epsilon} < \infty \quad \dots(10)$$

According to Bayes's theorem, the unconditional probability function for $(\underline{\epsilon})$ is as follows:

$$f(\underline{\epsilon}) = \int_0^\infty f(\underline{\epsilon}|\gamma)\, f(\gamma)\, d\gamma$$

$$f(\underline{\epsilon}) = \int_0^\infty \frac{1}{(2\,\pi\,\gamma\,\sigma^2)^{\frac{n}{2}}}\, e^{-\frac{1}{2\,\gamma\,\sigma^2}\,\underline{\epsilon}'\underline{\epsilon}} * \frac{\left(\frac{1}{2}\right)^{1/2}}{\sqrt{\pi}}\, \gamma^{-\frac{1}{2}-1}\, e^{-\frac{1}{2\gamma}}\, d\gamma$$

$$f(\underline{\epsilon}) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{(\sigma^2)^{\frac{n}{2}}(\pi)^{\frac{n+1}{2}}} \left[1 + \frac{1}{\sigma^2}\,\underline{\epsilon}'\underline{\epsilon}\right]^{-\left(\frac{n+1}{2}\right)} \quad \dots(11)$$

Equation (11) represents the probability function of $(\underline{\epsilon})$ and follows the multivariate Cauchy distribution. Since the vector of the response variable is a linear combination in terms of $(\underline{\epsilon})$, its probability function also follows the multivariate Cauchy distribution, with the following formula:

$$f(\underline{Y}) = C.\left[1 + \frac{1}{\sigma^2}\left(\underline{Y} - \lambda W\underline{Y} - X\underline{\beta}\right)'(I - \theta W)\left(\underline{Y} - \lambda W\underline{Y} - X\underline{\beta}\right)\right]^{-\left(\frac{n+1}{2}\right)} \quad \dots(12)$$

$$Whereas:\ C = \frac{\Gamma\left(\frac{n+1}{2}\right)\, |I - \theta W|^{\frac{1}{2}}}{(\sigma^2)^{\frac{n}{2}}(\pi)^{\frac{n+1}{2}}}$$

Assuming that $\alpha = (I - \lambda W)$ and making some mathematical simplifications, equation (12) becomes as follows:

$$f(\underline{Y}) = C^*.\left[1 + \frac{1}{\sigma^2}\left(\underline{Y} - \alpha^{-1}X\underline{\beta}\right)'\alpha'(I - \theta W)\alpha\left(\underline{Y} - \alpha^{-1}X\underline{\beta}\right)\right]^{-\left(\frac{n+1}{2}\right)} \dots(13)$$

While $C^* = \frac{\Gamma\left(\frac{n+1}{2}\right)\left|\alpha'(I-\theta W)\,\alpha\right|^{\frac{1}{2}}}{(\sigma^2)^{\frac{n}{2}}(\pi)^{\frac{n+1}{2}}}$

Equation (13) is described as follows:

$$\underline{Y} \sim MC\left(\alpha^{-1}X\underline{\beta}, \sigma^2\left(\alpha'(I - \theta W)\alpha\right)^{-1}\right)$$

5. Matrix of Spatial Adjacencies (Weights):

A spatial adjacency matrix (weights) is defined as a matrix of equal rows and columns and is built based on the adjacency between locations and the diagonal elements in it are equal to zero. Accordingly, in this research, a standard row weight matrix will be used, sometimes called the modified matrix, in which the sum of each row is equal to one, and in its calculation, it depends on the weights of the binary adjacencies according to the following formula: [1]

$$w_{ij}^{std} = \left|\begin{matrix} \frac{w_{ij}}{\sum w_{ij}} & i\ neignbor\ j & 0 < w_{ij}^{std} \leq 1 \\ 0 & & otherwise \end{matrix}\right| \dots(14)$$

6. Maximum Likelihood Method for Hybrid Spatial Model:

The maximum likelihood method of the spatial error model is concerned with estimating the parameter that explains the correlation between errors, while the spatial autoregressive model is concerned with estimating the parameter that reflects the spatial effects produced by the nature of the spatial correlation between the values of the response variable. [1]

$$L(\underline{Y}|\gamma,\sigma^2) = \frac{\left|\alpha'(I-\theta W)\alpha\right|^{\frac{1}{2}}}{(2\pi\gamma\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\gamma\sigma^2}\left(\underline{Y}-\alpha^{-1}X\underline{\beta}\right)'\alpha'(I-\theta W)\alpha\left(\underline{Y}-\alpha^{-1}X\underline{\beta}\right)} \quad \dots(15)$$

By taking the natural logarithm of both sides of equation (15) and finding the first partial derivative relative to the vector of parameters $\left(\underline{\beta}\right)$ and setting it equal to zero, the maximum likelihood estimator conditional on the variable (γ) is obtained as follows:

$$\ln L(\underline{Y}|\gamma,\sigma^2) = \ln\left|\alpha'(I-\theta W)\alpha\right|^{\frac{1}{2}} - \frac{n}{2}\ln(2\pi\gamma\sigma^2) - \frac{1}{2\gamma\sigma^2}\left(\underline{Y}-\alpha^{-1}X\underline{\beta}\right)'\alpha'(I-\theta W)\alpha\left(\underline{Y}-\alpha^{-1}X\underline{\beta}\right)$$

$$\frac{\partial \ln L(\underline{Y}|\gamma,\sigma^2)}{\partial \underline{\beta}} = -2X'(I-\theta W)\alpha\underline{Y} + 2X'(I-\theta W)X\underline{\beta}$$

$$\underline{\hat{\beta}} = (X'(I-\theta W)X)^{-1}X'(I-\theta W)(I-\lambda W)\underline{Y} \quad \dots(16)$$

$$\underline{\hat{\beta}}_{ML} = (X'(I-\theta W)X)^{-1}X'\underline{Y} - \lambda(X'(I-\theta W)X)^{-1}X'W\underline{Y} - \theta(X'(I-\theta W)X)^{-1}X'W\underline{Y} + \theta\lambda(X'(I-\theta W)X)^{-1}X'WW\underline{Y} \quad \dots(17)$$

Equation (17) represents the maximum likelihood estimator of the parameter of the hybrid spatial regression model function that follows the multivariate Cauchy distribution.

7. Lagrange Multiplier for SAR Model and SEM:

The Lagrange multiplier test is considered one of the important tests for detecting spatial dependence. This test determines the appropriate model for spatial models, whether it is a spatial autoregressive model or a spatial error model, by testing the following hypotheses: [3] [1]

$$H_0: \lambda = 0 \qquad V.S. \qquad H_1: \lambda \neq 0 \qquad \dots(18)$$

The mathematical formula for the test is as follows:

$$LM-SAR = \left(\frac{\frac{(\underline{\varepsilon}'W\underline{Y})}{\underline{\varepsilon}'\underline{\varepsilon}/n}}{d}\right)^2 \quad \dots(19)$$

$$Where: d = \frac{(WX\hat{\beta})'(I-X(X'X)^{-1}X)(WX\hat{\beta})}{\underline{\varepsilon}'\underline{\varepsilon}/n} + trace\,(W'W+WW)$$

When the null hypothesis defined by equation (18) is accepted, this means that there is no spatial dependence between the variables, while if the alternative hypothesis is accepted, this means that there is spatial dependence, then the appropriate model is the spatial autoregressive model.

As for the Lagrange multiplier for the (SEM), the hypothesis testing is as follows:

$$H_0: \theta = 0 \qquad V.S. \qquad H_1: \theta \neq 0 \qquad \dots(20)$$

This test shows the presence or absence of spatial dependence within the error limits. When the null hypothesis is rejected and the alternative hypothesis, which is the presence of spatial dependence, is accepted, the appropriate model is the (SEM), and the mathematical formula for the test is as follows:

$$LM-SEM = \left(\frac{\frac{(\underline{\varepsilon}'W\underline{\varepsilon})}{\underline{\varepsilon}'\underline{\varepsilon}/n}}{tr\,(W'W+WW)}\right)^2 \quad \dots(21)$$

Equations (19) and (21) are compared with the tabular chi-square value with one degree of freedom and a certain significance level.

8. Bayesian Methodology for Estimating a Hybrid Spatial Model Based on Initial Information Belonging to the known Probabilistic Family:

In this research, the parameter of the hybrid spatial model function that follows the multivariate Cauchy distribution was estimated based on initial information belonging to the known probability family. The initial distribution of the parameter of the hybrid spatial model function $(\underline{\beta})$ conditional on the variable (γ) is a multivariate normal distribution with the parameters $\left(\underline{\beta}_0 , {p_0}^{-1}\, \sigma^2\gamma\right)$ and the probability function is in the following form:

$$P\left(\underline{\beta}\middle|\sigma^2,\gamma\right) \propto \ e^{-\frac{1}{2\gamma\,\sigma^2}\left(\underline{\beta}-\underline{\beta}_0\right)'{p_0}^{-1}\left(\underline{\beta}-\underline{\beta}_0\right)} \quad \dots(22)$$

By combining the probability function defined in Equation (15) with the probability function defined in Equation (22), we obtain the kernel of the posterior probability distribution of $\left(\underline{\beta}\middle|\sigma^2,\gamma\right)$ as follows:

$$P\left(\underline{\beta}\middle|\underline{Y},\sigma^2,\gamma\right) \propto P\left(\underline{\beta}\middle|\sigma^2,\gamma\right)\, f\left(\underline{Y}\middle|\underline{\beta},\sigma^2,\gamma\right) \quad \dots(23)$$

$$P\left(\underline{\beta}\middle|\underline{Y},\sigma^2,\gamma\right) \propto e^{-\frac{1}{2\gamma\sigma^2}\left[\left(\underline{Y}-\alpha^{-1}X\underline{\beta}\right)'\alpha'(I-\theta W)\alpha\left(\underline{Y}-\alpha^{-1}X\underline{\beta}\right)+\left(\underline{\beta}-\underline{\beta}_0\right)'{p_0}^{-1}\left(\underline{\beta}-\underline{\beta}_0\right)\right]} \quad \dots(24)$$

Adding and subtracting the quantity $\left(\alpha^{-1}X\underline{\hat{\beta}}_{ML}\right)$ to the quadratic form $\left(\underline{Y}-\alpha^{-1}X\underline{\beta}\right)'\alpha'(I-\theta W)\alpha\left(\underline{Y}-\alpha^{-1}X\underline{\beta}\right)$ and making some mathematical simplifications, equation (24) becomes as follows:

$$P\left(\underline{\beta}\middle|\underline{Y},\sigma^2,\gamma\right) \propto e^{-\frac{1}{2\gamma\sigma^2}\left[\left(\underline{\beta}-\underline{\beta}_0\right)'{p_0}^{-1}\left(\underline{\beta}-\underline{\beta}_0\right)+\left(\underline{\beta}-\underline{\hat{\beta}}_{ML}\right)'X'(I-\theta W)X\left(\underline{\beta}-\underline{\hat{\beta}}_{ML}\right)\right]} \quad \dots(25)$$

The quadratic form defined in Equation (25) is similar to the quadratic form below: [8]

$$\left(\underline{z}-\underline{a}\right)'K\left(\underline{z}-\underline{a}\right)+\left(\underline{z}-\underline{b}\right)'L\left(\underline{z}-\underline{b}\right) = \left(\underline{z}-\underline{E}\right)'(K+L)\left(\underline{z}-\underline{E}\right) + \left(\underline{a}-\underline{b}\right)'K(K+L)^{-1}L\left(\underline{a}-\underline{b}\right) \quad \dots(26)$$

Where $(\underline{a},\underline{b},\underline{E},\underline{z})$ are vectors of dimension $(p\times 1)$ and (K,L) are matrices of dimension $(p\times p)$ and both are positive definite matrices, and $(\underline{E})$ defined in equation (26) is calculated as follows:

$$\underline{E} = (K+L)^{-1}\left(K\underline{a}+L\underline{b}\right) \quad \dots(27)$$

After similar the quadratic form defined in Equation (25) with the quadratic form defined in Equation (26), we obtain the posterior probability distribution of $\left(\underline{\beta}\middle|\underline{Y},\sigma^2,\gamma\right)$:

$$P\left(\underline{\beta}\middle|\underline{Y},\sigma^2,\gamma\right) = \frac{\left|{p_0}^{-1}+X'(I-\theta W)X\right|^{\frac{1}{2}}}{\left(2\pi\gamma\sigma^2\right)^{\frac{k}{2}}} * e^{-\frac{1}{2\gamma\sigma^2}\left[\left(\underline{\beta}-\underline{\hat{\beta}}_{bayes}\right)'{p_0}^{-1}+X'(I-\theta W)X\left(\underline{\beta}-\underline{\hat{\beta}}_{bayes}\right)\right]} \quad \dots(28)$$

$Where$:

$$\underline{\hat{\beta}_{bayes}} = \left({p_0}^{-1}+X'(I-\theta W)X\right)^{-1}\left({p_0}^{-1}\underline{\beta}_0 + X'(I-\theta W)X\,\underline{\hat{\beta}}_{ML}\right)$$

Since $\left(\underline{\hat{\beta}}_{ML}\right)$ was previously defined in Equation (17), and based on Bayes' theorem, the posterior probability distribution of $(\underline{\beta})$ is as follows:

$$P\left(\underline{\beta}\middle|\underline{Y},\sigma^2\right) = \int_0^\infty \left(\underline{\beta}\middle|\underline{Y},\sigma^2,\gamma\right) f(\gamma)\, d\gamma \quad \dots(29)$$

$$P\left(\underline{\beta}\middle|\underline{Y},\sigma^2\right) = \frac{\Gamma\left(\frac{k+1}{2}\right)\left|{p_0}^{-1}+X'(I-\theta W)X\right|^{\frac{1}{2}}}{\left(\sigma^2\right)^{\frac{k}{2}}(\pi)^{\frac{k+1}{2}}} * \left[1+\frac{1}{\sigma^2}\left(\underline{\beta}-\underline{\hat{\beta}_{bayes}}\right)'{p_0}^{-1} + X'(I-\theta W)X\left(\underline{\beta}-\underline{\hat{\beta}_{bayes}}\right)\right]^{-\left(\frac{k+1}{2}\right)} \quad \dots(30)$$

The probability function of $(\underline{\beta})$ defined in equation (30) represents a multivariate Cauchy distribution and can be described as follows:

$$\underline{\beta} \sim MC_k\left(\underline{\hat{\beta}_{bayes}}, \sigma^2\left({p_0}^{-1}+X'(I-\theta W)X\right)^{-1}\right)$$

Under the absolute loss function, represented by the median of equation (30), the Bayesian estimator of $(\underline{\beta})$ is as follows:

$$\underline{\hat{\beta}_{bayes}} = \left({p_0}^{-1}+X'(I-\theta W)X\right)^{-1}\left({p_0}^{-1}\underline{\beta}_0 + X'(I-\theta W)X\ \underline{\hat{\beta}}_{ML}\right) \quad \dots(31)$$

$$\underline{\hat{\beta}}_{ML} = \left(X'(I-\theta W)X\right)^{-1}X'\,\underline{Y} - \lambda\left(X'(I-\theta W)X\right)^{-1}X'W\,\underline{Y} - \theta\left(X'(I-\theta W)X\right)^{-1}X'W\,\underline{Y} + \theta\,\lambda\left(X'(I-\theta W)X\right)^{-1}X'W\,W\,\underline{Y}$$

9. Bayesian Prediction for Hybrid Spatial Model:

Predictive distribution represents the probability function of future observations (Y_f) conditional on a set of current observations(Y). If we have future observations (*f*) for all response variables, which represent the matrix(Y_f). Based on future observations, the hybrid spatial model is as follows: [9]

$$\underline{Y_f} = \lambda W_f \underline{Y_f} + X_f \underline{\beta} + \epsilon_f \quad \ldots(32)$$

Where $(\underline{Y_f})$ is a vector of future observations (*f*) with dimension $(n_f \times 1)$ and (W_f) is a matrix of future adjacencies (weights) with dimension$(n_f \times n_f)$. And (X_f) is a matrix with dimension $(n_f \times (k+1))$. And $(\underline{\epsilon_f})$ is the future random error vector with dimension $(n_f \times 1)$, and assuming that the error vector $(\underline{\epsilon_f})$ follows a multivariate Cauchy distribution with parameters $\left(\underline{0}, \sigma^2 \left(\alpha_f{}'(I_f - \theta W_f)\alpha_f\right)^{-1}\right)$, and that $(\underline{Y_f})$ is a linear combination in terms of future errors, therefore $(\underline{Y_f})$ follows a multivariate Cauchy distribution with parameters$\left(\alpha_f{}^{-1} X_f \underline{\beta}, \sigma^2 \left(\alpha_f{}'(I_f - \theta W_f)\alpha_f\right)^{-1}\right)$.

$$f(\underline{Y_f}) = C^*{}_f . \left[1 + \frac{1}{\sigma^2}\left(\underline{Y_f} - \alpha_f{}^{-1} X_f \underline{\beta}\right)' \alpha_f{}'(I_f - \theta W_f)\alpha_f \left(\underline{Y_f} - \alpha_f{}^{-1} X_f \underline{\beta}\right)\right]^{-\left(\frac{n_f+1}{2}\right)} \quad \ldots(33)$$

$$Wereas: C^*{}_f = \frac{\Gamma\left(\frac{n_f+1}{2}\right)\left|\alpha_f{}'(I_f - \theta W_f)\alpha_f\right|^{\frac{1}{2}}}{(\sigma^2)^{\frac{n_f}{2}}(\pi)^{\frac{n_f+1}{2}}}$$

After merging the posterior probability distribution of the parameter vector $(\underline{\beta})$ conditional on the random variable (γ) defined by equation (28) with the probability function of $(\underline{Y_f})$ conditional on the random variable (γ) and defined by the following equation (34):

$$f\left(\underline{Y_f}\middle|\gamma, \underline{\beta}, \sigma^2\right) = \frac{\left|\alpha_f{}'(I_f - \theta W_f)\alpha_f\right|^{\frac{1}{2}}}{(2\pi\sigma^2\gamma)^{\frac{n_f}{2}}} e^{-\frac{1}{2\gamma\sigma^2}\left(\left(\underline{Y_f} - \alpha_f{}^{-1} X_f \underline{\beta}\right)' \alpha_f{}'(I_f - \theta W_f)\alpha_f \left(\underline{Y_f} - \alpha_f{}^{-1} X_f \underline{\beta}\right)\right)} \quad \ldots(34)$$

We obtain the predictive distribution of $\underline{Y_f}$ conditional on the random variable (γ) as follows:

$$f(\underline{Y_f}|\underline{Y}, \gamma) = \int_{\underline{\beta}} f\left(\underline{Y_f}\middle|\gamma, \underline{\beta}, \sigma^2\right) P\left(\underline{\beta}\middle|\underline{Y}, \sigma^2, \gamma\right) d\underline{\beta}$$

$$f(\underline{Y_f}|\underline{Y}, \gamma) = \frac{\left|\alpha_f{}'(I_f - \theta W_f)\alpha_f\right|^{\frac{1}{2}} |p_0{}^{-1} + X'(I - \theta W)X|^{\frac{1}{2}}}{(2\pi\sigma^2\gamma)^{\frac{n_f}{2}} |p_0{}^{-1} + X'(I - \theta W)X + X'{}_f(I_f - \theta W_f)X_f|^{\frac{1}{2}}} * e^{-\frac{1}{2\sigma^2\gamma}\left[\left(\underline{Y_f} - \alpha_f{}^{-1} X_f \underline{\hat{\beta}}_{Bayes}\right)' \alpha_f{}'(I_f - \theta W_f)\alpha_f \left(\underline{Y_f} - \alpha_f{}^{-1} X_f \underline{\hat{\beta}}_{Bayes}\right)\right]} \quad \ldots(35)$$

Since $(\underline{\hat{\beta}}_{Bayes})$ was previously defined in equation (31), and after performing the integration process for equation (35) relative to the random variable (γ), we obtain the prediction distribution of future observations conditional on the vector of current observations as follows:

$$f(\underline{Y_f}|\underline{Y}) = \int_0^\infty f(\underline{Y_f}|\underline{Y}, \gamma) f(\gamma)\, d\gamma$$

$$f(\underline{Y_f}|\underline{Y}, \gamma) = \frac{\Gamma\left(\frac{n_f+1}{2}\right)\left|\alpha_f{}'(I_f - \theta W_f)\alpha_f\right|^{\frac{1}{2}} |p_0{}^{-1} + X'(I - \theta W)X|^{\frac{1}{2}}}{(\sigma^2)^{\frac{n_f}{2}}(\pi)^{\frac{n_f+1}{2}} |p_0{}^{-1} + X'(I - \theta W)X + X'{}_f(I_f - \theta W_f)X_f|^{\frac{1}{2}}} \left[1 + \frac{1}{\sigma^2}\left(\underline{Y_f} - \alpha_f{}^{-1} X_f \underline{\hat{\beta}}_{Bayes}\right)' \alpha_f{}'(I_f - \theta W_f)\alpha_f \left(\underline{Y_f} - \alpha_f{}^{-1} X_f \underline{\hat{\beta}}_{Bayes}\right)\right]^{-\left(\frac{n_f+1}{2}\right)} \quad (36)$$

The prediction probability distribution defined in equation (36) is not one of the common probability distributions, but it is a proper distribution. Using the properties of mathematical expectation, the Bayesian prediction of the vector of future observations of the response variable $(\underline{Y})$ under the quadratic loss function is as follows:

$$E(\underline{Y_f}|\underline{Y}) = \int_0^\infty \int_{\underline{\beta}} E\left(\underline{Y_f}\middle|\underline{\beta}, \sigma^2, \gamma\right) \times P\left(\underline{\beta}\middle|\underline{Y}, \gamma, \sigma^2\right) f(\gamma)\, d\underline{\beta}\, d\gamma = \alpha_f{}^{-1} X_f \underline{\hat{\beta}}_{Bayes} \quad \ldots(37)$$

Since $(\underline{\hat{\beta}}_{Bayes})$ was previously defined in equation (31).

10. Application Side:

In this study, data on the number of people suffering from bronchial asthma in Baghdad Governorate for the year (2004) and the factors affecting it were statistically analyzed based on (Matlab R2022a) programming.

10.1 Defining Basic Variables and Initializing the Data:

The research data represents the number of patients with bronchial asthma in Baghdad Governorate for the year (2004) as a response variable (Y) and the variables affecting it as explanatory variables, which are lead (X_1), carbon monoxide (X_2), sulfur dioxide (X_3) and total suspended particles (X_4). Table (1) shows the monthly concentrations of air pollutants and the monthly numbers of patients with bronchial asthma, which were collected by the Ministry of Environment / Planning and Technical Follow-up Department / Air Quality Section. Figure (1) shows the behaviour of the variable number of patients with bronchial asthma in Baghdad for the year (2004).

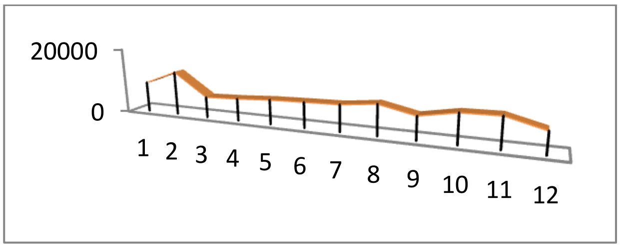

Figure 1: Behavior of data on the number of people with bronchial asthma in Baghdad Governorate.

To determine the suitability of the data, the (Matlab R2022a) program was used to test the model used. The study data was tested through the goodness of fit test, and the extracted Kolmogorov-Smirnov test value (0.1984) for this model was smaller than the tabular value ($d_{tab}(0.05,12) = 0.2420$), which indicates that the data is suitable for the model used.

To find out whether there is a spatial dependence of the variable of patients with bronchial asthma in Baghdad Governorate, the Lagrange multiplier tests defined in equations (19) and (21) were used, depending on the standard class weight matrix defined in equation (14). The values of the two tests were as follows:

Table 1: Lagrange Multiplier Test Value for the SAR model and SEM.

Test	**Test values based on standard class weight matrix**	$\chi^2(0.95,1)$
$LM - SAR$	5.2347	**3.841**
$LM - SEM$	4.4421	**3.841**

From Table 1, we notice that the value of the Lagrange multiplier test for the spatial autoregressive model and the spatial error is greater than the tabular value under the significance level (∝=0.05), which indicates the existence of a spatial correlation between the observations of the variable of patients with bronchial asthma, i.e. ($\lambda \neq 0$), in addition to the existence of a spatial correlation between the observations of the error, i.e. ($\theta \neq 0$).

The data of bronchial asthma patients were divided into two parts, the first consisting of a sample of size ($n = 10$) to be used in the estimation process, and the last two observations were used to conduct the Bayesian prediction process. The parameter vector $(\underline{\beta})$ for the hybrid spatial model of bronchial asthma patient's data of size ($n = 10$) was estimated using the traditional method of choosing the initial values ($\underline{\beta_0}$) from it, and these values were as follows:

$$\underline{\beta_0} = \begin{bmatrix} 160 \\ 700 \\ -12500 \\ 20 \end{bmatrix} \quad ; \quad \sigma = 2034.26$$

$$p_0 = \begin{bmatrix} 6.8 & -0.3 & 0.006 & -68.8 \\ -0.3 & 0.153 & -7.2e-05 & -12.4 \\ 0.006 & -7.2e-05 & 0.0003 & -0.47 \\ -68.8 & -12.4 & -0.47 & 6889 \end{bmatrix}$$

The parameter (λ) is found by maximizing the concentrated likelihood function defined as:

$$|I - \lambda W| = \prod_{i=1}^{10} (1 - \lambda w_i)$$

$$Ln|I - \lambda W| = \sum_{i=1}^{10} \ln(1 - \lambda w_i)$$

After obtaining the characteristic roots of the standard class weight matrix, the solution can be reached using the non-linear optimization method. After substituting the estimators of the parameters of the hybrid spatial model, what is called the concentrated likelihood function is produced, this contains only the spatial effect parameter. The formula for the concentrated likelihood function is as follows:

$$Lc = -5\ln\left(\frac{(\epsilon_0 - \lambda\epsilon_L)'(\epsilon_0 - \lambda\epsilon_L)}{10}\right) + \sum_{i=1}^{10}\ln(1-\lambda w_i) \quad \dots (38)$$

Using the iterative method, we obtain the value of the spatial effect parameter, which belongs to the interval $\left[\frac{1}{\text{Smaller characteristic root}}, \frac{1}{\text{Larger characteristic root}}\right]$, and thus the value of the spatial effect parameter is (λ=0.905).

As for the parameter (θ), it is found by maximizing the concentrated likelihood function as follows:

$$|I - \theta W| = \prod_{i=1}^{10}(1-\theta w_i)$$

$$Ln|I - \theta W| = \sum_{i=1}^{10}\ln(1-\theta w_i)$$

$$Lc = -5\ln\left(\frac{(\epsilon)'(\epsilon)}{10}\right) + \sum_{i=1}^{10}\ln(1-\theta w_i) \quad \dots (39)$$

Using the iterative method of equation (39) and performing the same steps to find the parameter (λ), the value of the parameter (θ = 1.567).

10.2 Bayesian Methodology for Estimating Parameter of a Hybrid Spatial Model Function that Follows a Multivariate Cauchy Distribution:

In this section, we find an estimate of the vector parameters of the hybrid spatial model function using the Bayesian method when the initial information is available and belongs to the known probability family. The following table shows the value of the estimate of the parameter of the hybrid spatial regression model function that follows a multivariate Cauchy distribution.

Table 2: Estimated Values of the Vector Parameters of the Hybrid Spatial Model Function.

Variables	$\hat{\beta}_{bayes}$	**p-value**
X_1	162.0969	**0.492742891**
X_2	725.7295	**0.045214565**
X_3	-11983.8	**0.0345687121**
X_4	17.82202	**0.006658727**
λ	٠.٩٠٥	
θ	١.٥٦٧	

We note from Table (2) that the p-values of the variables carbon monoxide (X_2), sulfur dioxide (X_3), and total suspended particles (X_4) are less than the significance level (∝=0.05), meaning that they have significant effects on the incidence of bronchial asthma in Baghdad Governorate during the year (2004). Table (3) shows the values of the parameters of the general regression model, assuming that there is no spatial dependence among the observations of bronchial asthma.

Table 3: Estimated Values of the Vector Parameters of the General Regression Model Function

Variables	$\hat{\beta}_{bayes}$	**p-value**
X_1	16٣.٠٥٤١	**0.٥١٠٢٤١١٥٢**
X_2	72٧.٣٤١٩	**0.04٩٩٨٤٥١٢**
X_3	-119٩٩.١	**0.0٣٩٨٥٤٢١٢**
X_4	1٨.٠١٢٥٤	**0.00٢٥٤١٢٣٦**

We note from Table (3) that the p-values of the variables carbon monoxide (X_2), sulfur dioxide (X_3), and total suspended particles (X_4) are less than the significance level (∝=0.05), meaning that they have significant effects on the incidence of bronchial asthma in Baghdad Governorate during the year (2004). Table (4) shows the values of the standards (MAE and MSE) for the estimated hybrid spatial model and the estimated general regression model.

Table 4: Values of the Standards (MAE and MSE) for the Estimated Models

	Estimated hybrid spatial model	**Estimated general regression model**
MAE	٢٥.١١٢٤٧	**٢٩.٤١٤٥2**
MSE	**٢٤.٩٥٤١9**	**٢٩.١٠٢٣3**

From Table (4), we note the superiority of the estimated hybrid spatial model over the estimated general regression model, based on the criteria of mean square error (MSE) and mean absolute error (MAE).

The following figure shows the real and estimated values of data for bronchial asthma patients based on the estimated hybrid spatial model.

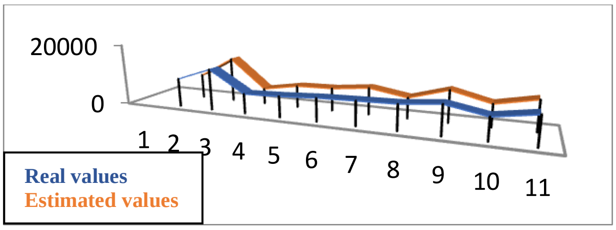

Figure 2: Behavior of real and estimated values of bronchial asthma patient's data in Baghdad Governorate.

Figure (2) shows that the estimated values of the vector of bronchial asthma patients have the same pattern as the real values, indicating that the estimated hybrid spatial model was appropriate for the study data.

10.3 Bayesian Prediction:

In this section, the future values of data for patients with bronchial asthma will be predicted based on the Bayes estimator for the hybrid spatial model defined in Table (2), and the predictive value represents the predictive mean defined in Equation (37). Since:

$$\hat{\underline{Y}}_f = \alpha_f^{-1} X_f \hat{\underline{\beta}}_{Bayes}$$

Table 5: Real and Predictive Values

	Real values	Predictive values
١	9112	٩١٠٥
2	6325	٦٤٢٥

The following figure shows the behaviour of true and predicted values.

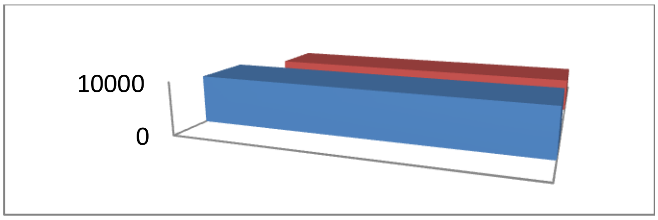

Figure 3: Behavior of real and predictive values for bronchial asthma patients

11. Conclusions:

Through the results reached, a set of points were concluded that can be summarized as follows:

- The posterior probability distribution of the parameter vector $\underline{\beta}$ is a multivariate Cauchy distribution with parameters $\left(\hat{\underline{\beta}}_{bayes}, \sigma^2 \left(p_0^{-1} + X'(I - \theta W)X\right)^{-1}\right)$.
- The prediction distribution of the hybrid spatial model does not belong to the family of known probability distributions, but it is a proper distribution.
- Emergence of spatial dependence between observations of the number of people with bronchial asthma in Baghdad Governorate for the year (2004).
- The estimated hybrid spatial model outperformed the general regression model estimated based on the mean square error (MSE) and mean absolute error (MAE).
- Carbon monoxide, sulfur dioxide and total suspended particles are variables that have significant effects on the number of people suffering from bronchial asthma in Baghdad Governorate for the year (2004).

References:

[1] Abo-al Shaeer, M. J. and Al-Sarraf, N. M. (2018)" Linear Regression: Insights from the rule to summit", Abdul Salam Printing Press, First edition, Al- Rafidain University College, Iraq.

[2] Al-Hashmi, A. M.A and AL-Shammari, M. A. K. H. (2021)" Spatial analysis of air pollution and heavy elements in urban centres in Wasit governorate", Al-Qadissiyah Journal for Humanities Sciences, http://dx.doi.org/10.13140/RG.2.2.12501.65763

[3] Ali, O. A. and Hadi, S. Q. (2014)" Spatial Regression Models Estimation For the Poverty Rates in the Districts of Iraq in 2012", Journal of Economics and Administration Sciences, 20(79),p.p.337-351.

[4] Al-Saraifi, A. N. A. (2022)" The environmental effects of the falling dust over the Safwan district and the center of Basra Governorate", journal of Diyala for Human Research, Vol. 4 No. 94.

[5] Atikah, N., Widodo, B., Rahardjo, S., Kholifia, N., & Afifah, D. L. (2021)" The efficiency of Spatial Durbin Model (SDM) parameters estimation on advertisement tax revenue in Malang City", Journal of Physics: Conference Series (Vol. 1821, No. 1, p. 012012). IOP Publishing.

[6] Bechmann, A. L. C. F. (2016), "Estimating an Inverse Gamma distribution", >stat>arXiv:1605.01019v2.

[7] Bouhlel, N., & Rousseau, D. (2022)," A generic formula and some special cases for the Kullback–Leibler divergence between central multivariate Cauchy distributions". Entropy, 24(6), 838.

[8] Box G. P. and Tiao G. C. (1973) "Bayesian Inference in Statistical Analysis" Addison Wesley Publishing Company, Inc. London, U.K.

[9] Judge, G. G., Griffiths, W. E, Lutkepohl, H and Lee, T. (1985),“ The Theory and practice of econometrics”, John Wiley and Sons, U.S.A.

[10] Lee, C. Y. (2015), “Fast simulated annealing with a multivariate Cauchy distribution and the configuration’s initial temperature”. Journal of the Korean Physical Society, 66, 1457-1466.

[11] Lee, H. Y., Park, H. J., & Kim, H. M. (2014). A clarification of the Cauchy distribution. *Communications for Statistical Applications and Methods, 21*(2), 183-191.

[12] Zhu, Y., Han, X. and Chen, Y. (2020)” Bayesian estimation and model selection of threshold spatial Durbin model”, Economics Letters, vol. 188, p.108956.

[13] https://www.sciencedirect.com/topics/computer-science/cauchy-distribution