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The Log (t+1) Weibull Distribution: A New Flexible Model for Slow peaking survival data

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ABSTRACT

This paper aims to develop a new distribution, known as the Log (t+1) WD distribution, as a modification of the basic Weibull distribution by replacing the time variable t with the logarithmic variable log (1+t). This aims to achieve a more accurate and realistic representation of survival data with nonlinear behavior. The statistical properties of the new distribution were analyzed and compared with the Weibull distribution in terms of density functions, survival, and hazard functions. Trade-off measures (AIC, AICc, BIC, HQ, SHWTZ) and the mean square error (MSE) criterion were also used to measure the goodness of fit. The results showed that the Log (t+1) WD distribution exhibits a smoother behavior of the probability density function, starting from zero at t=0 and gradually rising to a peak representing the most probable time of the event, unlike the Weibull, which may exhibit a sharp jump at the beginning. Furthermore, the proposed distribution offers high flexibility in representing survival data over time, leading to increased allocation.

1. Introduction

The survival data must be analysed using basic statistical methods in medical and epidemiological fields, such as those related to infection or its consequences. Several possible distributions have been used to model this data, most notably the Weibull, Gompertz, and normalized logarithmic distributions, thus allowing for specific control over these phenomena. However, despite their popularity in representing survival data, these distributions are not suitable for complex phenomena characterized by a long period of quiescence before the onset of multivariate growth. Therefore, the need arises to develop a more suitable possible distribution for easier handling of such data. This paper proposes a new distribution, called the Log (t+1) WD distribution, which relies on transforming the traditional Weibull function via a time-logarithmic component. This enables it to represent more complex survival behaviours, especially in the early stages characterized by stability. The paper reviews the mathematical properties of the new distribution, comparing it behaviourally and numerically with traditional distributions in terms of density, hazard, and survival functions. It also conducts simulations to estimate its parameters and verify its effectiveness in representing biological data with greater accuracy and flexibility. Many researchers have used the Weibull distribution in analysing survival data and have developed the distribution using different methods. Klakattawi (2022) proposed a new distribution

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called the Alpha Power Kumaraswamy Weibull Distribution, a generalization of the Weibull distribution aimed at increasing the model's flexibility in analysing survival data [1]. Li et al. (2024), This study used the parametric model of the Weibull distribution to analyse the survival of cancer patients, where the effect of treatment was estimated using the hazard ratio [2]. Rangoli et al. (2025) presented the Modified Weibull Distribution, capable of representing several forms of the risk function, such as increasing, decreasing, and constant, as well as the Bathtub Shape. The distribution was tested using medical data [10]. Alsulami et al. (2025) proposed a new generalization of the Weibull distribution, examining its statistical properties and estimating its parameters using several methods such as maximum likelihood (MLE) and least squares. The model was also applied to real-world data [11]. Gong et al. (2025) introduced the Transformed Inverse Weibull Distribution to enhance the model's capacity to depict lifespan data. Multiple techniques were employed to estimate parameters, including maximum likelihood and Bayesian estimation [12]. Ahmad et al. (2025) investigated the Extended Weibull Distribution with progressive censoring in fault data analysis. The distribution parameters were estimated using several statistical methods, including Bayesian and Monte Carlo simulations [13].

In this paper we developed a new probability distribution for modelling data that suffer from slow peaking survival depend on Weibull as basic distribution, and estimate the parameter of suggested distribution by Maximum Likelihood Method to exam the distribution behaviour by Monte-Carlo simulation experiments and under real data set , and extract the basic properties for this distribution.

2. Weibull distribution

The Weibull distribution is used in survival data analysis, survival analysis, and failure time modelling [3], [4]

The probability density functions of the Weibull distribution as follows: [5], [6], [7]

$$f(t,k,\lambda) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-\left(\frac{t}{\lambda}\right)^{K}} \quad ; \quad t, k, \lambda > 0 \tag{1}$$

Where λ Scale parameter, k shape parameter

The cumulative function as follows: [14], [15]

$$F(t,k,\lambda) = 1 - e^{-\left(\frac{t}{\lambda}\right)^{K}} \tag{2}$$

$$R(t,k,\lambda) = e^{-\left(\frac{t}{\lambda}\right)^{K}} \tag{3}$$

$$E(T) = \lambda\Gamma\left(\frac{1}{k}+1\right) \tag{4}$$

$$Var(T) = \lambda^2\left[\Gamma\left(\frac{2}{k}+1\right) - \Gamma^2\left(\frac{1}{k}+1\right)\right] \tag{5}$$

$$Mode(T) = \lambda\Gamma\left(\frac{K-1}{k}\right)^{\frac{1}{K}} \tag{6}$$

$$Median(T) = \lambda\,(\ln(2))^{\frac{1}{K}} \tag{7}$$

$$E(T^n) = \lambda^n\,\Gamma\left(1+\frac{n}{k}\right) \tag{8}$$

Knowing that the moment generating function is not defined in closed form except for special cases (e.g., k=1)

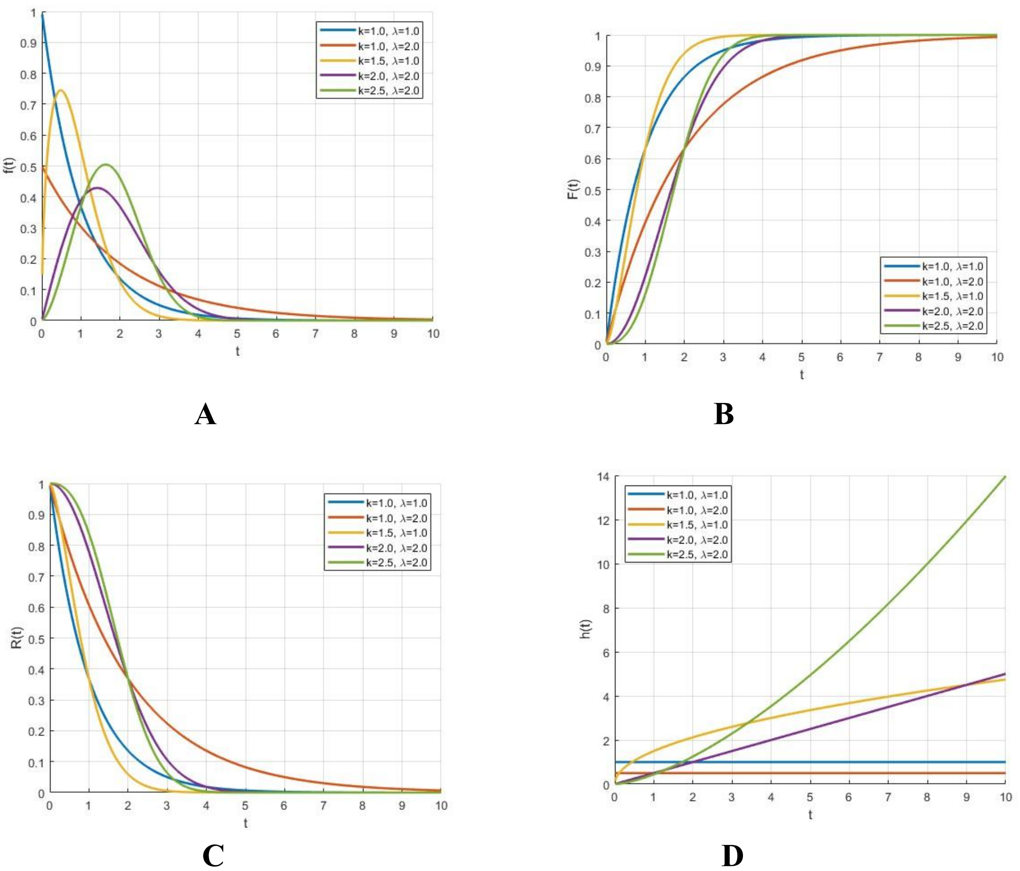


Figure 1: A: Probability density function B: Cumulative density function C: Reliably function D: Hazard function curves of Weibull distribution

3. Survival data analysis

Survival Analysis (SA) is an important branch of statistics that deals with time-to-event outcomes. Because of its importance in many diverse areas of application, adaptation of machine and deep learning techniques to SA has received increased interest in recent years [2]. Survival Analysis typically focuses on time to event data. In the most general sense, it for positive valued random variables, such as time to death, time to relapse of a disease, length of a contract, duration of a policy, money paid by health insurance, viral load measurements, time to finishing a master thesis, Typically, survival data are not fully observed, but rather are censored [8], [9], [16]

4. The Log (t+1) probability density

In many survival analysis applications, time t is used directly in probabilistic model building. However, some data types, such as survival times for chronic diseases or delayed failure times in engineering systems, exhibit distinctive behavior: the hazard rate is very low initially, then gradually increases, and may increase more rapidly over time. When t is used directly in some conventional probability distributions, this behavior may not be accurately represented, especially when t is close to zero, as the hazard function may become unstable or start with unrealistic values. To overcome this problem, a time transformation based on the natural logarithm can be used in the form U = log (1 + T), where T ≥ 0, and therefore U ≥ 0. This transformation has several important properties; it ensures that the function is defined at time zero because log (1 + 0) = 0. It also slows down the growth of time in the early stages because the function log (1 + t) grows slowly when the values of t are small, while allowing for a relatively faster increase over time. Therefore, this transformation is useful in modeling survival data where the risk rate is initially low and then gradually increases over time.

Let $U = log(1 + T)$;T ≥ 0 U ≥ 0, Then,

$$T = e^{U} - 1 \rightarrow u = log(1 + t)$$

Then $\frac{du}{dt} = \frac{1}{1+t}$

$$\therefore \mathrm{f_T(t)} = \mathrm{f_U(u)}\left|\frac{du}{dt}\right| \quad (9)$$

By substituting $u = log(1 + t)$ equation (9), we obtain

$$\mathrm{f_T(t)} = \mathrm{f_U}(log(1 + t))\frac{1}{1+t} \quad ; \mathrm{t} > 0 \quad (10)$$

Where,

$$\mathrm{t} < \mathrm{T} < t + \Delta\mathrm{t} \Leftrightarrow \log(1 + t) < U < \log(1 + t + \Delta\mathrm{t}) \quad (11)$$

Because $\log(1 + t)$ is increasing function then,

$$\mathrm{f_U(u)} = \lim_{\Delta u \to 0} \frac{\mathrm{p_r(u<U< u+\Delta u)}}{\Delta u} \quad ; \mathrm{u} \geq 0 \quad (12)$$

and $\Delta u = \log(1 + t + \Delta t) - \log(1 + t)$, then when $\Delta t \rightarrow 0$ then,

$$\Delta u \sim \frac{\Delta t}{1+t} \quad (13)$$

Then by using U density,

$$\mathrm{P}(\log(1 + t) < \mathrm{U} < \log(1 + t + \Delta t)) \approx \mathrm{f_U}(\log(1 + t))\Delta u \quad (24)$$

Then we obtain:

$$\mathrm{f_T(t)} = \mathrm{f_U}(\log(1 + t))\frac{1}{1+t} \quad ; \mathrm{t} > 0 \quad (14)$$

5. The Log (t+1) Weibull Distribution:

If U=Log (t+1) follow Weibull Distribution, then the density of Log (1+t) as follows:

$$f(t) = \frac{k}{\lambda(1+t)}\left(\frac{\log(1+t)}{\lambda}\right)^{k-1} e^{-\left(\frac{\log(1+t)}{\lambda}\right)^{k}} \quad ; \quad t > 0 \quad , \quad (15)$$

Where t is survival time or the time until the event occurs, λ Scale parameter where directly affects the spread of the distribution along the time axis. The larger it is, the more concentrated the distribution is, meaning that the event tends to occur after a longer period of observation. This translates into higher median survival, such as in cases characterized by slow disease progression or delayed onset of symptoms. Conversely, small values of λ result in density concentration around small values of t which indicating an early onset of the event and a lower median survival. k Shape parameter which represent the backbone represents the elasticity of the distribution and its associated hazard behaviour. This parameter controls the slope of the density and the behaviour of the hazard function over time.

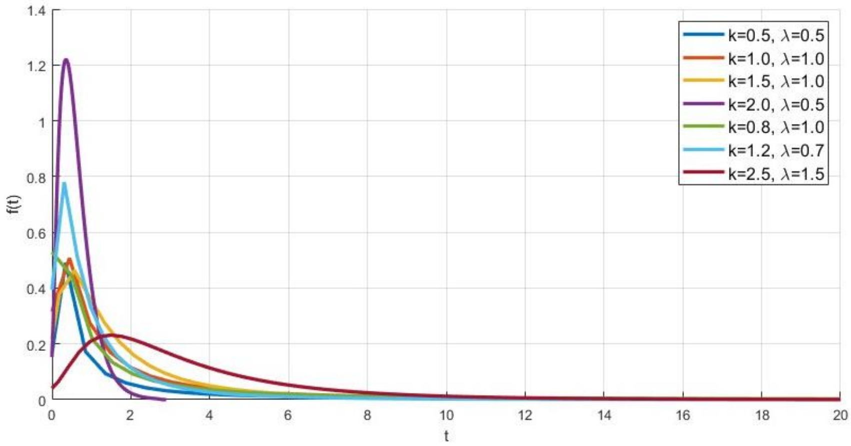


Figure (2) Log (t+1) WD Survival Distribution PDF

Figure (2) illustrates the interval between time and risk. At t=0, the probability density value is always zero regardless of the values of k and λ, this is because the distribution is defined using log (1+t) which is equal to zero when t=0. This prevents any illogical jump at the beginning of time and gives realistic behavior for systems where the event does not occur immediately. At t=0, there is a significant difference, but it varies considerably for the value of k: when k > 1, it is close to t=0, then it has a probability initially, then it becomes more likely. When k < 1, the probability lags and increases over time. When k = 1, you get an exponential-logarithmic distribution.

The cumulative distribution function as follows,

$$F(t) = 1 - e^{-\left(\frac{\log(1+t)}{\lambda}\right)^{k}} \quad (16)$$

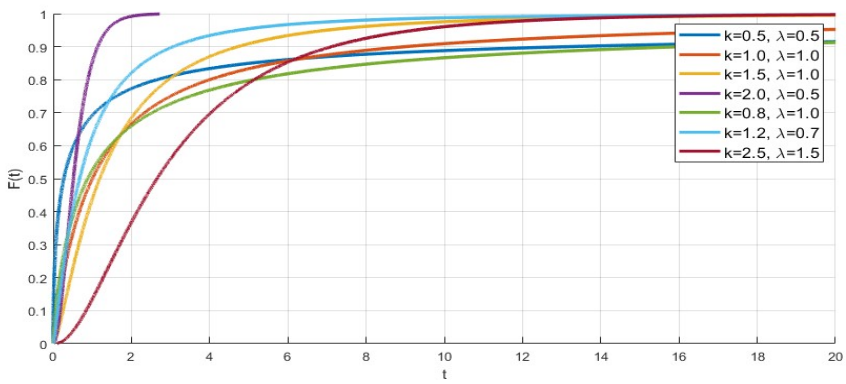


Figure (3) the CDF curves of The Log (t+1) WD Survival Distribution

Then the survival function as follows,

$$S(t) = e^{-\left(\frac{\log(1+t)}{\lambda}\right)^{k}} \qquad (17)$$

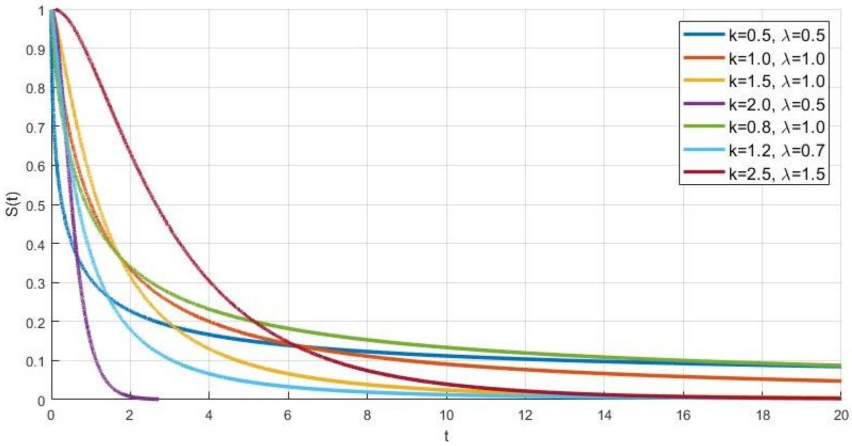


Figure (4) the Survival curves of The Log (t+1) WD Survival Distribution

The hazard function as follows,

$$h(t) = \frac{k}{\lambda(1+t)}\left(\frac{\log(1+t)}{\lambda}\right)^{k-1} \qquad (18)$$

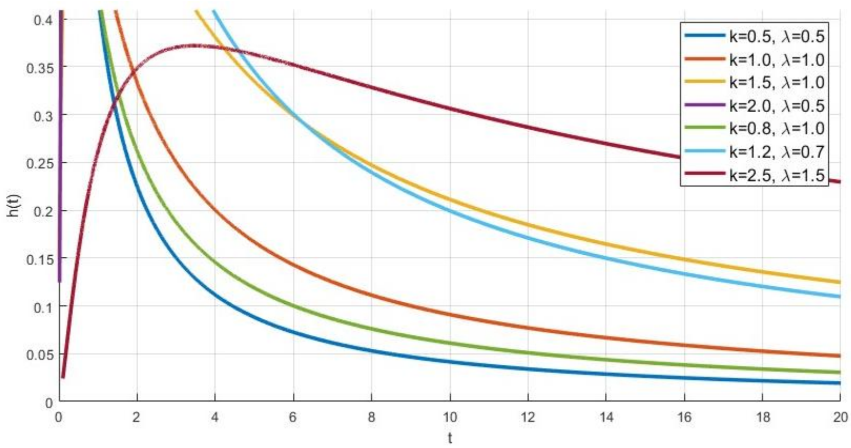


Figure (5) the Hazard curves of The Log (t+1) WD Survival Distribution

Figure (5) showed the hazard function curves h (t) that explain the probability of an event occurring provided the object or system survives until time t. All curves start from zero at t=0, a logical behavior that reflects the fact that the hazard cannot be non-zero at the initial moment, especially with the factor log (1+t)) that initially adjusts the growth rate. The curves then diverge significantly depending on the values of the parameters k and λ.

• When k<1 we observe that the hazard function starts low and decreases steadily, indicating a decreasing hazard over time, a behavior common in systems that experience early failure and subsequently stabilise.

• When k=1, the function is quasi-stationary or tends to be constant, reflecting a constant risk that does not change over time.

• When k>1, as in the purple and red curves, the function starts at zero, then gradually rises until it reaches a clear hazard peak, and then begins to decline. This behavior is known as "non-monotonic hazard" and is observed in many biological processes or complex systems that deteriorate after a period of good performance.

The last curve in dark red representing the case k=2.5, λ=1.5, clearly shows how the hazard function can start low, rise to a peak, and then gradually stabilize.

The moment generating function as follows,

$$m_T(s) = \sum_{r=0}^{\infty} S^r \sum_{n=r}^{\infty} \mathrm{S}(\mathrm{n},\mathrm{r}) \frac{\lambda^n \Gamma\left(\frac{n}{k}+1\right)}{n!} \quad (19)$$

Where $\mathrm{S}(\mathrm{r},\mathrm{n})$ are the Stirling numbers of the second kind, which represent the number of ways to partition a set of r elements into n non-empty sub-sets.

Proof:

While $\mathrm{U} = \log(1 + \mathrm{T})$;$\mathrm{T} \geq 0$ $\mathrm{U} \geq 0$, Then, $\mathrm{T} = e^{\mathrm{U}} - 1$

$$m_T(s) = E(e^{sT}) = E\left(e^{s(e^U - 1)}\right) = e^{-s} E\left(e^{se^U}\right)$$

By using Taylor Expansion we get:

$$e^{se^U} = \sum_{r=0}^{\infty} \frac{S^r e^{rU}}{r!} \text{, then,}$$

$$m_T(s) = \sum_{r=0}^{\infty} \frac{S^r}{r!} E(T^r)$$

And $E(U^n) = \lambda^n \Gamma\left(\frac{n}{k} + 1\right)$, then

$$(e^U - 1)^r = r! \sum_{n=r}^{\infty} S(n,r) \frac{U^n}{n!}$$

$$\therefore E(T^r) = r! \sum_{n=r}^{\infty} S(n,r) \frac{E(U^n)}{n!}$$

$$= r! \sum_{n=r}^{\infty} S(n,r) \frac{\lambda^n \Gamma\left(\frac{n}{k}+1\right)}{n!}$$

$$\therefore m_T(s) = \sum_{r=0}^{\infty} S^r \sum_{n=r}^{\infty} S(n,r) \frac{\lambda^n \Gamma\left(\frac{n}{k}+1\right)}{n!}$$

And,

$$E\big(Log(t+1)\big) = \lambda \Gamma\left(1 + \frac{1}{k}\right) \quad (20)$$

$$E\left(\big(Log(t+1)\big)^2\right) = \lambda^2 \left[\Gamma\left(1 + \frac{2}{k}\right)\right] \quad (21)$$

$$Var\big(Log(t+1)\big) = \lambda^2 \left[\Gamma\left(\frac{2}{k} + 1\right)\right] - \Gamma^2\left(1 + \frac{1}{k}\right) \quad (22)$$

6. Maximum Likelihood estimation

The Likelihood function for log (t+1) Weibull as follows,

$$L = \prod_{i=1}^{n} f(t_i, k, \lambda)$$

$$\left(\frac{k}{\lambda(1+t_i)}\right)^n \prod_{i=1}^{n} \left(\frac{\log(1+t_i)}{\lambda}\right)^{k-1} e^{-\sum_{i=1}^{n}\left(\frac{\log(1+t_i)}{\lambda}\right)^k} \quad (17)$$

$$l = Log(L) = nLog(k) - n\lambda Log(1 + t_i) + (k-1) \sum_{i=1}^{n} \log\left(\frac{\log(1+t_i)}{\lambda}\right) - \sum_{i=1}^{n} \left(\frac{\log(1+t_i)}{\lambda}\right)^k \quad (23)$$

We derive equation (23) for each of the distribution parameters k and λ , and set the derivative equal to zero to obtain the maximum Likelihood values for the distribution parameters as follows:

$$\frac{dl}{dk} = \frac{n}{k} + \sum_{i=1}^{n} \log\left(\frac{\log(1+t_i)}{\lambda}\right) - \sum_{i=1}^{n} \left(\frac{\log(1+t_i)}{\lambda}\right)^{k} \log\left(\frac{\log(1+t_i)}{\lambda}\right) \quad (24)$$

$$\frac{dl}{d\lambda} = -nLog(1+t_i) + \frac{(k-1)n}{\lambda} + \sum_{i=1}^{n} \left(\frac{\log(1+t_i)}{\lambda}\right)^{k-1} \log\left(\frac{1+t_i}{\lambda^2}\right) \quad (25)$$

The equations (24), (25) solved numerically using MatLab Program

7. Simulation Study

In this section, the behaviour of the proposed distribution was tested using Monte-Carlo simulation experiments by choosing to implement three experiments that included three initial values for the parameters of the proposed distribution (k=2, 2.5, 0.5; λ=0.5, 1.5) by choosing four sample sizes (n=50, 75, 100, 150). The parameters of the proposed distribution were estimated using the maximum likelihood method. The mean square error was extracted for each of the probability density function and the survival function of the distribution. The data were generated using the inverse of the cumulative probability density function as follows,

$$t = e^{-\lambda \log(1-u)^{\frac{1}{k}} - 1} \quad (26)$$

Where u is a random variable that follows a uniform distribution in the interval (0, 1), True_pdf the theoretical pdf, pdf _MLE is maximum likelihood estimate for pdf , True_S is theoretical survival function , S_MLE is S_MLE maximum likelihood estimate for survival function ,

The results as the following:

First Experiment:

Table 1: Estimates of PDF, Survival function and ML MSE for Log (t+1) WD under $k = 2, \lambda = 0.5$

n	t	True_pdf	pdf _MLE	MSE_{pdf}	True_S	S_MLE	MSE_S
50	0.01000	0.07878	0.04074	0.00145	0.99960	0.99982	0.00000
	0.19602	0.05326	0.07408	0.00043	0.87971	0.89072	0.00012
	0.38203	0.23214	0.40181	0.02879	0.65787	0.64826	0.00009
	0.56805	0.02157	0.15773	0.01854	0.44513	0.40493	0.00162
	0.75406	0.72474	0.76121	0.00133	0.28278	0.22648	0.00317
	0.94008	0.47165	0.43549	0.00131	0.17259	0.11698	0.00309
	1.12609	0.29153	0.22726	0.00413	0.10272	0.05705	0.00209
	1.31211	0.17459	0.11134	0.00400	0.06020	0.02670	0.00112
	1.49812	0.10257	0.05220	0.00254	0.03498	0.01213	0.00052
	1.68414	0.05960	0.02373	0.00129	0.02025	0.00540	0.00022
75	0.01000	0.07878	0.05411	0.00061	0.99960	0.99974	0.00000
	0.21572	0.10347	0.02151	0.00672	0.85845	0.87611	0.00031
	0.42145	0.20685	0.20590	0.00000	0.60975	0.63577	0.00068
	0.62717	0.92748	0.96642	0.00152	0.38749	0.40870	0.00045
	0.83289	0.60900	0.64807	0.00153	0.23028	0.24304	0.00016
	1.03862	0.36735	0.39324	0.00067	0.13142	0.13744	0.00004
	1.24434	0.21103	0.22461	0.00018	0.07323	0.07525	0.00000
	1.45007	0.11784	0.12357	0.00003	0.04027	0.04038	0.00000
	1.65579	0.06477	0.06642	0.00000	0.02201	0.02141	0.00000
	1.86151	0.03532	0.03522	0.00000	0.01202	0.01129	0.00000
100	0.01000	0.07878	0.10473	0.00067	0.99960	0.99946	0.00000
	0.31458	0.23403	0.27165	0.00142	0.74138	0.71980	0.00047
	0.61916	0.94044	0.91075	0.00088	0.39497	0.37453	0.00042
	0.92374	0.49099	0.46218	0.00083	0.18045	0.16982	0.00011
	1.22832	0.22061	0.20606	0.00021	0.07669	0.07260	0.00002
	1.53289	0.09273	0.08717	0.00003	0.03159	0.03041	0.00000
	1.83747	0.03793	0.03625	0.00000	0.01290	0.01272	0.00000
	2.14205	0.01540	0.01508	0.00000	0.00528	0.00537	0.00000
	2.44663	0.00629	0.00634	0.00000	0.00219	0.00230	0.00000
	2.75121	0.00259	0.00271	0.00000	0.00092	0.00101	0.00000
150	0.01000	0.07878	0.02265	0.00315	0.99960	0.99990	0.00000
	0.22637	0.12691	0.03367	0.00869	0.84657	0.88382	0.00139
	0.44274	0.18749	0.35205	0.02708	0.58425	0.60965	0.00064
	0.65911	0.87565	0.03590	0.02568	0.35869	0.34509	0.00018
	0.87548	0.55148	0.60354	0.00271	0.20559	0.16887	0.00135
	1.09185	0.31943	0.29656	0.00052	0.11317	0.07433	0.00151
	1.30822	0.17651	0.13000	0.00216	0.06088	0.03029	0.00094
	1.52460	0.09499	0.05265	0.00179	0.03237	0.01167	0.00043
	1.74097	0.05042	0.02016	0.00092	0.01713	0.00432	0.00016
	1.95734	0.02661	0.00742	0.00037	0.00907	0.00155	0.00006

Table 2: Criteria's of comparison between Weibull and Log (t+1) WD for $k = 2, \lambda = 0.5$

n	Dis.	AIC	AICc	BIC	HQ	SHWTZ
50	Log (t+1) WD	19.26775	19.52307	23.0918	20.72397	0.015112
	Weibull	22.11608	22.3714	25.94013	23.5723	0.026441
75	Log (t+1) WD	57.77004	57.9367	62.40501	59.62073	0.01854
	Weibull	57.24768	57.41434	61.88265	59.09837	0.008419
100	Log (t+1) WD	66.14541	66.26913	71.35575	68.25413	0.005926
	Weibull	73.35981	73.48352	78.57015	75.46853	0.011584
150	Log (t+1) WD	64.73953	64.82117	70.7608	67.18578	0.004961
	Weibull	67.21944	67.30108	73.24071	69.66569	0.005086

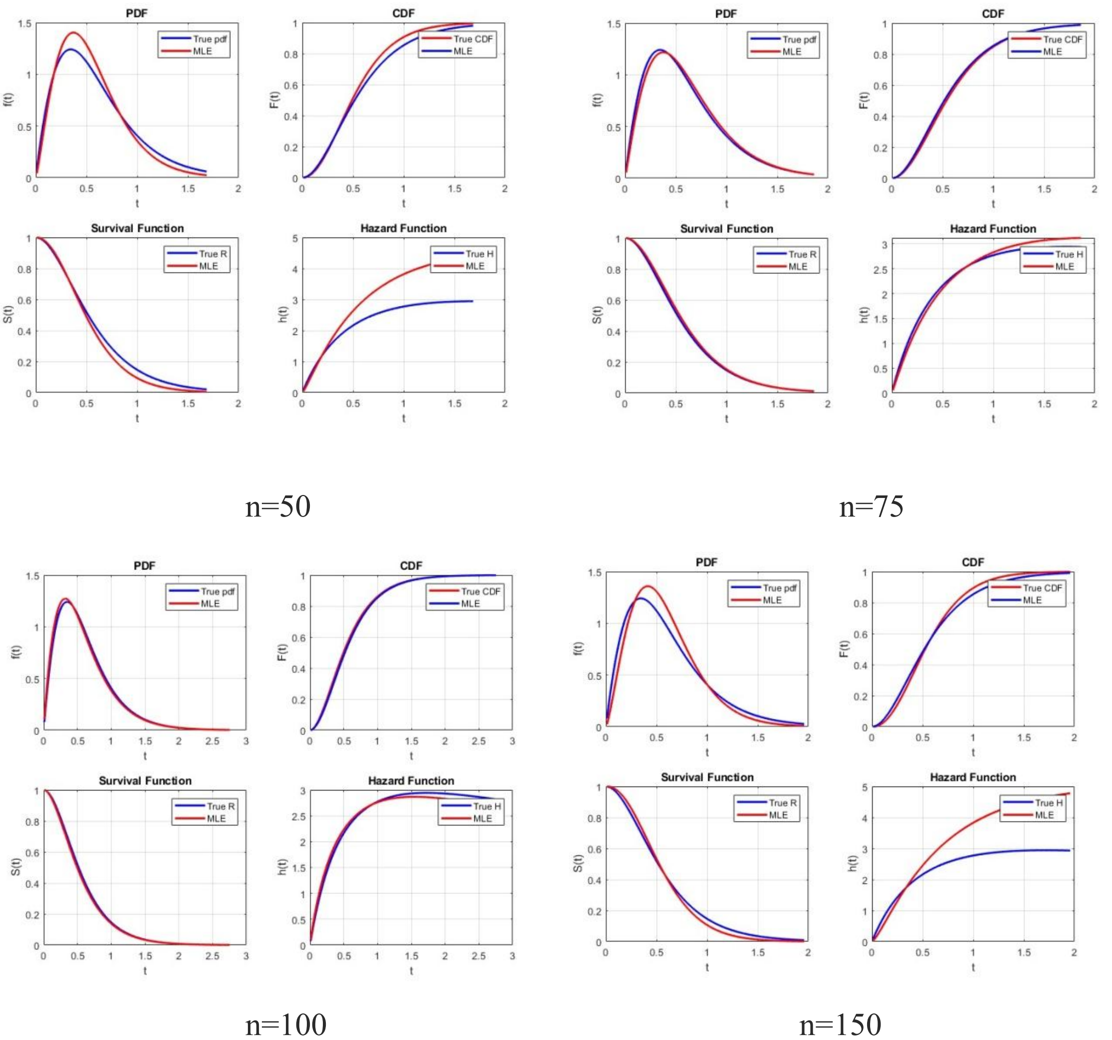


Figure1: PDF, CDF, R, H for Log (t+1) WD under $k = 2, \lambda = 0.5$ for all sample sizes

Second Experiment:

Table 3: Estimates of PDF, Survival function and ML MSE for Log (t+1) WD under $k = 2.5, \lambda = 1.5$

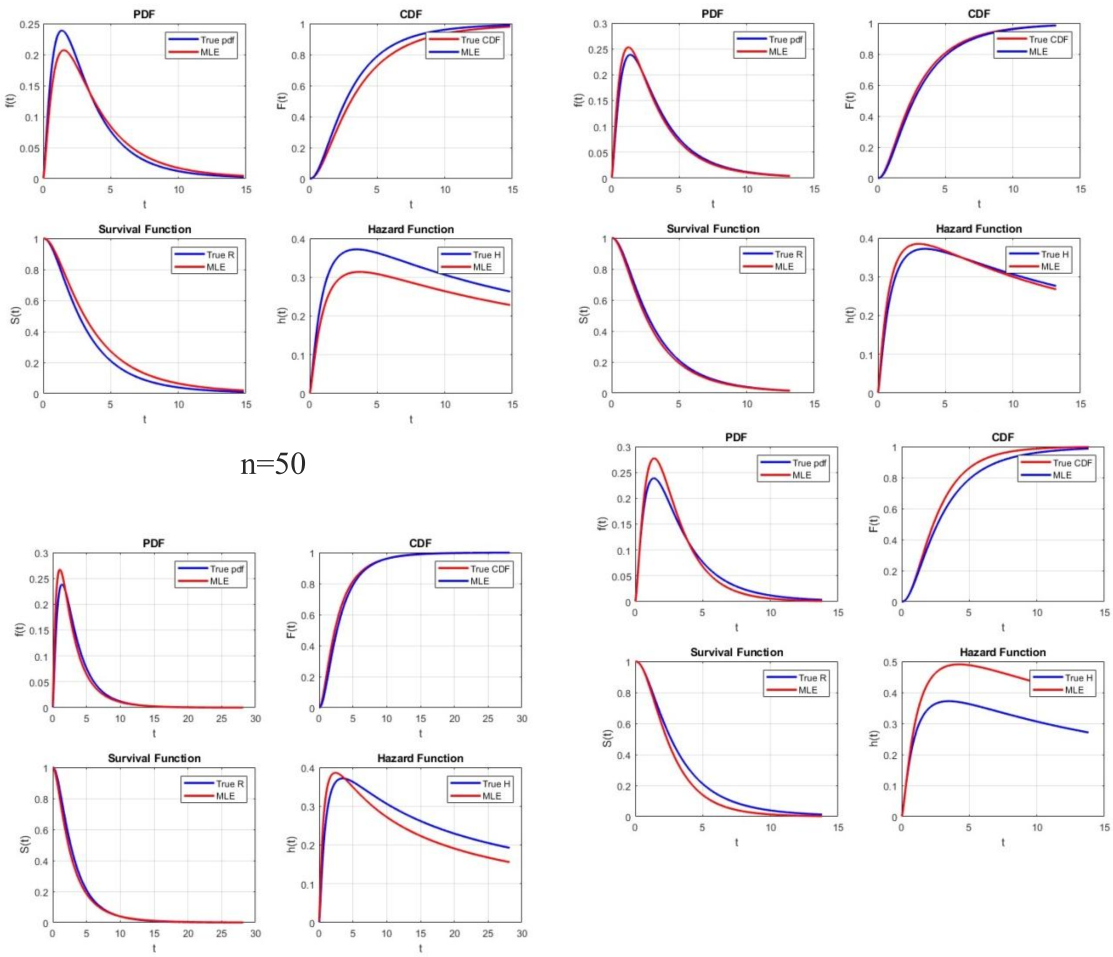
n	t	True_pdf	pdf _MLE	MSE_{pdf}	True_S	S_MLE	MSE_S

50	0.01000	0.00089	0.00059	0.00000	1.00000	1.00000	0.00000
	1.65659	0.23418	0.20635	0.00077	0.71005	0.75774	0.00227
	3.30318	0.14611	0.14470	0.00000	0.39313	0.46251	0.00481
	4.94977	0.07775	0.08536	0.00006	0.21412	0.27659	0.00390
	6.59635	0.04120	0.04960	0.00007	0.11949	0.16820	0.00237
	8.24294	0.02240	0.02931	0.00005	0.06881	0.10482	0.00130
	9.88953	0.01257	0.01774	0.00003	0.04088	0.06696	0.00068
	11.53612	0.00727	0.01101	0.00001	0.02498	0.04377	0.00035
	13.18271	0.00433	0.00699	0.00001	0.01566	0.02923	0.00018
	14.82930	0.00264	0.00454	0.00000	0.01005	0.01989	0.00010
75	0.01000	0.00089	0.00158	0.00000	1.00000	0.99999	0.00000
	1.47524	0.23811	0.24918	0.00012	0.75292	0.72417	0.00083
	2.94048	0.16627	0.16131	0.00002	0.44973	0.41939	0.00092
	4.40573	0.09611	0.08981	0.00004	0.26124	0.23996	0.00045
	5.87097	0.05438	0.04983	0.00002	0.15392	0.14066	0.00018
	7.33621	0.03122	0.02835	0.00001	0.09288	0.08499	0.00006
	8.80145	0.01835	0.01662	0.00000	0.05748	0.05289	0.00002
	10.26669	0.01106	0.01004	0.00000	0.03643	0.03381	0.00001
	11.73193	0.00682	0.00623	0.00000	0.02360	0.02214	0.00000
	13.19718	0.00431	0.00396	0.00000	0.01560	0.01481	0.00000
100	0.01000	0.00089	0.00359	0.00001	1.00000	0.99998	0.00000
	3.13946	0.15502	0.13917	0.00025	0.41777	0.36543	0.00274
	6.26891	0.04667	0.03935	0.00005	0.13386	0.12055	0.00018
	9.39837	0.01488	0.01311	0.00000	0.04760	0.04687	0.00000
	12.52783	0.00530	0.00506	0.00000	0.01881	0.02068	0.00000
	15.65729	0.00208	0.00219	0.00000	0.00810	0.01004	0.00000
	18.78674	0.00089	0.00104	0.00000	0.00375	0.00524	0.00000
	21.91620	0.00040	0.00052	0.00000	0.00184	0.00290	0.00000
	25.04566	0.00019	0.00028	0.00000	0.00095	0.00168	0.00000
	28.17511	0.00010	0.00016	0.00000	0.00051	0.00101	0.00000
150	0.01000	0.00089	0.00053	0.00000	1.00000	1.00000	0.00000
	1.54437	0.23700	0.27551	0.00148	0.73650	0.70530	0.00097
	3.07875	0.15841	0.17046	0.00015	0.42729	0.35498	0.00523
	4.61312	0.08866	0.08224	0.00004	0.24209	0.16785	0.00551
	6.14750	0.04889	0.03814	0.00012	0.13966	0.07975	0.00359
	7.68187	0.02748	0.01786	0.00009	0.08275	0.03880	0.00193
	9.21624	0.01586	0.00857	0.00005	0.05040	0.01943	0.00096
	10.75062	0.00940	0.00423	0.00003	0.03149	0.01001	0.00046
	12.28499	0.00572	0.00215	0.00001	0.02014	0.00531	0.00022
	13.81936	0.00357	0.00112	0.00001	0.01316	0.00288	0.00011

Table 4: Criteria's of comparison between Weibull and Log (t+1) WD for $k = 2.5, \lambda = 1.5$

n	Dis.	AIC	AICc	BIC	HQ	SHWTZ
50	Log (t+1) WD	239.1286	239.3839	242.9527	240.5848	0.000343
	Weibull	239.6714	239.9268	243.4955	241.1277	0.000774
75	Log (t+1) WD	323.1667	323.3334	327.8017	325.0174	0.000203

	Weibull	326.1658	326.3325	330.8008	328.0165	0.000616
100	Log (t+1) WD	425.139	425.2627	430.3494	427.2477	0.000234
	Weibull	440.712	440.8357	445.9223	442.8207	0.000306
150	Log (t+1) WD	581.6865	581.7682	587.7078	584.1328	0.000187
	Weibull	598.2454	598.327	604.2666	600.6916	0.000306



n=50

n=100

n=150

Figure 2: PDF, CDF, R, H for Log (t+1) WD under $k = 2.5, \lambda = 1.5$ for all sample sizes

Third Experiment:

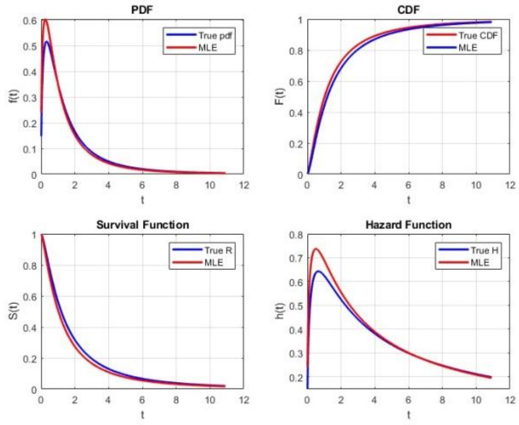
Table 5: Estimates of PDF, Survival function and ML MSE for Log (t+1) WD under $k = 0.5, \lambda = 0.5$

n	t	True_pdf	pdf _MLE	MSE_{pdf}	True_S	S_MLE	MSE_S
50	0.01000	0.14800	0.23964	0.00840	0.99901	0.99829	0.00000
	1.22169	0.29562	0.28739	0.00007	0.49006	0.43760	0.00275
	2.43338	0.12330	0.11039	0.00017	0.25410	0.21738	0.00135
	3.64508	0.05966	0.05138	0.00007	0.14908	0.12513	0.00057
	4.85677	0.03247	0.02737	0.00003	0.09537	0.07936	0.00026
	6.06846	0.01926	0.01604	0.00001	0.06490	0.05384	0.00012
	7.28015	0.01219	0.01007	0.00000	0.04627	0.03838	0.00006

	8.49184	0.00810	0.00667	0.00000	0.03418	0.02841	0.00003
	9.70353	0.00561	0.00461	0.00000	0.02599	0.02167	0.00002
	10.91523	0.00401	0.00330	0.00000	0.02023	0.01694	0.00001
75	0.01000	0.14800	0.08914	0.00346	0.99901	0.99944	0.00000
	1.56697	0.22716	0.24219	0.00023	0.40038	0.42570	0.00064
	3.12393	0.08017	0.08754	0.00005	0.18518	0.19263	0.00006
	4.68090	0.03526	0.03793	0.00001	0.10132	0.10150	0.00000
	6.23786	0.01800	0.01887	0.00000	0.06175	0.05940	0.00001
	7.79483	0.01019	0.01037	0.00000	0.04053	0.03745	0.00001
	9.35179	0.00622	0.00614	0.00000	0.02807	0.02495	0.00001
	10.90876	0.00402	0.00384	0.00000	0.02026	0.01735	0.00001
	12.46573	0.00271	0.00252	0.00000	0.01510	0.01249	0.00001
	14.02269	0.00190	0.00171	0.00000	0.01156	0.00924	0.00001
100	0.01000	0.14800	0.07236	0.00572	0.99901	0.99955	0.00000
	2.30996	0.13384	0.15012	0.00027	0.26995	0.35152	0.00665
	4.60993	0.03647	0.04781	0.00013	0.10386	0.15138	0.00226
	6.90989	0.01393	0.01990	0.00004	0.05109	0.07940	0.00080
	9.20986	0.00649	0.00980	0.00001	0.02897	0.04697	0.00032
	11.50982	0.00344	0.00540	0.00000	0.01803	0.03012	0.00015
	13.80979	0.00199	0.00323	0.00000	0.01197	0.02047	0.00007
	16.10975	0.00123	0.00205	0.00000	0.00835	0.01453	0.00004
	18.40971	0.00081	0.00136	0.00000	0.00605	0.01067	0.00002
	20.70968	0.00055	0.00094	0.00000	0.00452	0.00806	0.00001
150	0.01000	0.14800	0.08047	0.00456	0.99901	0.99949	0.00000
	4.09531	0.04703	0.05913	0.00015	0.12520	0.17854	0.00284
	8.18061	0.00896	0.01318	0.00002	0.03684	0.05968	0.00052
	12.26592	0.00285	0.00457	0.00000	0.01566	0.02729	0.00014
	16.35122	0.00118	0.00200	0.00000	0.00806	0.01476	0.00004
	20.43653	0.00057	0.00102	0.00000	0.00467	0.00887	0.00002
	24.52183	0.00031	0.00057	0.00000	0.00294	0.00574	0.00001
	28.60714	0.00018	0.00034	0.00000	0.00196	0.00391	0.00000
	32.69245	0.00011	0.00022	0.00000	0.00137	0.00278	0.00000
	36.77775	0.00007	0.00015	0.00000	0.00099	0.00204	0.00000

Table 6: Criteria's of comparison between Weibull and Log (t+1) WD for $k = 0.5, \lambda = 0.5$

n	Dis.	AIC	AICc	BIC	HQ	SHWTZ
50	Log (t+1) WD	160.7574	161.0128	164.5815	162.2137	0.003019
	Weibull	164.3048	164.5601	168.1289	165.761	0.004406
75	Log (t+1) WD	260.2629	260.4295	264.8978	262.1136	0.001375
	Weibull	265.4586	265.6253	270.0936	267.3093	0.001698
100	Log (t+1) WD	388.0523	388.176	393.2627	390.161	0.000418
	Weibull	405.7429	405.8666	410.9533	407.8516	0.001476
150	Log (t+1) WD	494.0757	494.1574	500.097	496.522	0.001491
	Weibull	497.8147	497.8963	503.836	500.2609	0.00153



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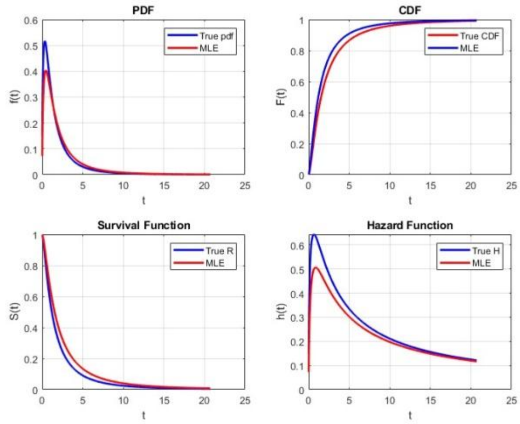
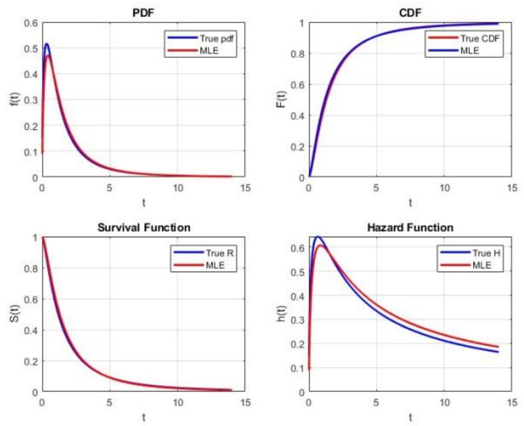


n=75



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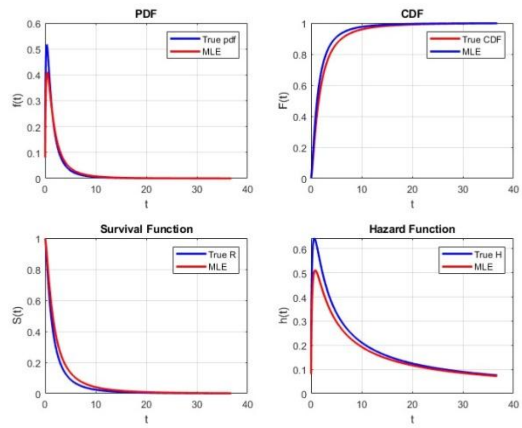

n=150

Figure3: PDF, CDF, R, H for Log (t+1) WD under $k = 0.5, \lambda = 0.5$ for all sample sizes

The experimental results show that the Log (t+1) WD distribution exhibits a behaviour that differs from the Weibull distribution in terms of the shape of the probability density function and its evolution over time. While the Weibull density function starts with a positive value at t=0, especially when k>1, and may exhibit an unrealistic jump in some short-term situations, the Log (t+1) WD, thanks to replacing time with log(1+t), always starts from zero at t=0. This gives it a smoother and more realistic behaviour, as the probability of an event occurring at the initial moment is virtually zero in most biological phenomena. After this point, the Log (t+1) WD density gradually increases until it reaches a peak representing the most probable time of the event, and then begins a slow decline. The location of this peak and the shape of the curve depend on the values of the parameters k and λ. When 1 > K, it undergoes a series of changes indicating that the initial risk is high, and this is followed by a decrease. When 1 < K, the light is considered late, and the nucleus has a long tail, which leads to a significant increase in many. However, the Weibull distribution shows a more recent change around the constitution and suffers weak beginning, which is minimally necessary. These differences in density shape are reflected in the numerical results; the Log (t+1) WD showed a lower MSE and greater stability in large samples, and had lower AIC and BIC values than the Weibull distribution in most cases, particularly in the second experiment (k = 2.5, λ = 1.5), where the density rose slowly, and then reached a clear peak before gradually declining. The Weibull, however, exhibited a faster and less elastic slope. Therefore, it can be argued that the Log (t+1) WD provides a more realistic description of phenomena characterized by an initial delay in the occurrence of an event followed by a subsequent acceleration, achieving a balance between mathematical convenience and physiological significance of temporal behaviour, and outperforming the original Weibull in representing survival systems with time-varying risk.

Simulation experiments also show that the Log (t+1) WD distribution has a high ability to represent the behaviour of survival data, which is characterized by a slow slope and gradient at the beginning and then accelerates later. It is clear that the estimated survival function (S_MLE) values closely approximate the true values (True_S), with the mean squared error (MSE) decreasing as the sample argument increases. In the first case (k=2, lect=0.5), the proposed distribution closely matches the conventional Weibull distribution, particularly at large enrolment sizes. In the second case (k=2.5, lect=1.5), the proposed distribution significantly outperforms the Weibull distribution, recording the lowest non-Weibull values for the differential criteria (AIC, BIC, HQ, SHWTZ). This indicates its accuracy in representing situations where the risk initially increases and then gradually decreases, i.e., systems with non-monotonic hazard behavior. In the third case (k=1.2, λ=0.5), despite the slight increase in differences across small samples, Log (t+1) WD maintained a limited superiority, demonstrating its flexibility in representing quasi-static risks. The results confirm that using log (1+t) in the distribution structure makes the behavior at t=0 more realistic and prevents irrational density jumps.

4. Real Data set:

Simulation experiments showed that the best results were in the first experiment (k=2, λ=0.5), as it showed high accuracy in the estimates of the PDF, survival function and hazard function, in addition to the superiority of the new distribution over the traditional Weibull distribution in different comparison criteria. Accordingly, these values were used to estimate the distribution functions for real data representing the survival times of pancreatic cancer patients witch represent the survival times of (150) patients with pancreatic cancer, measured in days, and are real data taken from the Iraqi Ministry of Health / Babylon Health Department / Babylon Oncology Center for the period 2022–2024. They represent the time from the beginning of diagnosis or follow-up until the occurrence of the event (death or end of study). Very small values indicate early deaths after diagnosis, which is common in pancreatic cancer due to its aggressive nature and the difficulty of early detection, while large values represent patients who responded to treatment or were diagnosed at less advanced stages, resulting in a longer survival time as shown in Table (8) as follows:

First, the data were tested for the Log (1+t) Weibull distribution, as shown in the following Table (8):

Table 8: Lifetime data in days for pancreatic cancer patients

48.41	4.38	6.33	25.35	4.60	6.36	37.53	18.29	60.96	14.38
28.57	13.21	4.59	26.13	0.23	1.46	8.88	2.26	27.77	1.29
6.67	30.69	10.42	14.95	41.21	1.50	4.17	2.17	29.95	0.08
47.78	12.55	17.95	26.12	10.57	5.67	1.5	53.1	4.38	15.03
17.18	4.15	6.51	16.03	11.79	59.13	3.07	19.07	0.60	3.29
10.02	17.32	41.96	3.52	17.52	2.93	41.48	0.64	5.50	1.12
10.33	20.48	0.61	4.87	14.55	0.26	15.46	7.74	11.71	7.69
1.32	4.14	8.23	3.67	43.81	21.92	11.06	8.64	6.01	0.79
1.46	7.21	52.15	7.20	43.91	25.01	16.28	20.73	26.76	33.11
14.03	11.62	12.08	0.6	30.7	6.47	8.7	4.39	4.82	0.43
12.27	10.44	8.38	14.65	15.39	9.50	19.91	3.10	51.09	13.15
3.64	7.51	12.78	6.24	24.55	15.96	0.74	1.28	9.29	8.78
44.44	5.37	13.05	16.22	17.19	0.75	12.7	8.50	28.06	16.95
23.12	1.60	20.01	7.4	12.95	45.3	2.63	17.74	3.29	6.05
1.83	0.84	2.08	17.41	2.09	33.12	1.95	0.91	8.06	2.56

Table 9: Criterions of distributions fitting

Dist.	AIC	BIC	HQ	Kolmogorov-Smirnov (K-S) test (P-value)	χ^2 Test (P-value)
Weibull	1098.29001	1104.31128	-1087.84375	0.41123(0.013)	2.427(0.016)
Log(1+t) Weibull	1097.13679	1101.22228	-1099.33400	0.97995(0.543)	13.947(0.787)

Table (9) shows that the Log(1+t) Weibull distribution outperforms the Wiebull distribution on all measures of fit. Log(1+t) Weibull demonstrates better fiting to the data due to lower AIC, BIC, and HQ values. The Kolmogorov–Smirnov test showed that the Weibull distribution was an incorrect fit to the data at a significance level of 0.05 because the p-value (0.013) was less than 0.05, while the Log(1+t) Weibull distribution was a good fit with a p-value (0.543), which is greater than 0.05. Similarly, the chi-square goodness-of-fit test confirmed this result, with the p-value (0.016) for the Weibull distribution rejecting the null hypothesis, while (0.787) for the Log(1+t) Weibull distribution was a good fit.

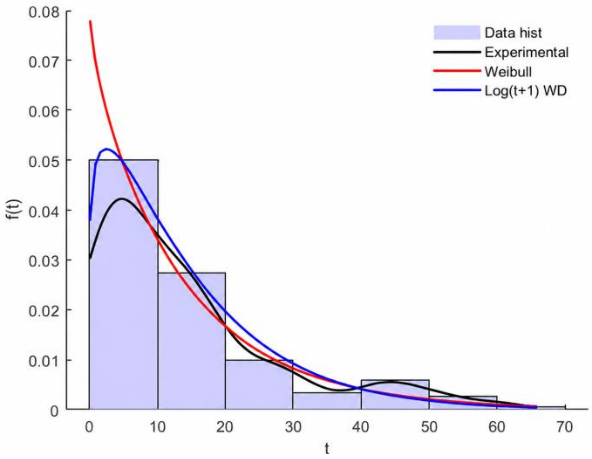

Figure 4: Experimental, Weibull, and Log (t+1) WD p.d.f curves under real data set

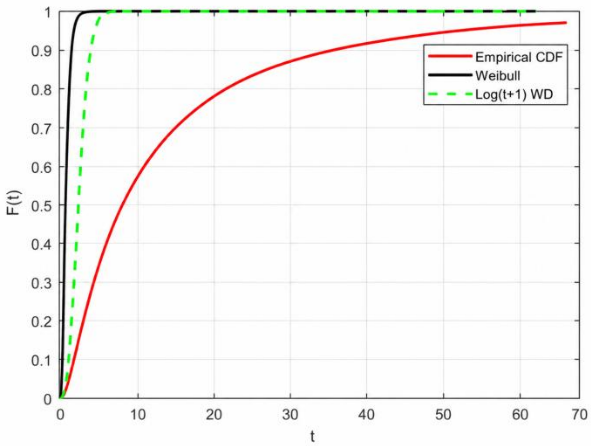

Figure 5: Experimental, Weibull, and Log (t+1) WD CDF curves under real data set

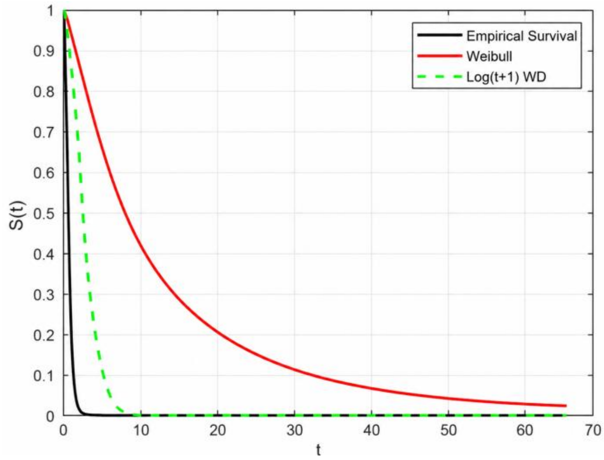

Figure 6: Experimental, Weibull, and Log (t+1) WD Survival curves under real data set

Figure (4) illustrates that the Weibull distribution provides a satisfactory fit to the experimental data at lower values; however, its performance gradually deteriorates in the upper-tail region, where noticeable deviations from the observed data emerge. also, the Log (t+1) Weibull distribution demonstrates a superior ability to capture the tail behavior, yielding a closer agreement with the experimental observations at higher values and providing an overall more accurate representation of the distribution's upper range. This suggests that the Log (t+1) WD distribution is the more suitable option for representing data with a long tail. Figure 5 shows that the Log (t+1) WD distribution fits the real data better, especially at the long tail, indicating that it can be more accurate in representing data than the Weibull distribution. Figure 5 shows that the Weibull distribution is well-defined at the beginning of the data and then becomes completely decoded at the ends, which does not accurately represent the data. However, the distribution best fits the experimental data. Figure 6 shows the survival function curves at the same rate.

Conclusions:

The results indicate that the proposed survival distribution exhibits an initial delay followed by a gradual increase in its rate over time. As time progresses, it provides a better representation of the survival data than the conventional Weibull distribution, particularly in situations where the event status is not completely determined at the initial time point ((t = 0)). This suggests that the proposed model is more effective in capturing delayed event dynamics and temporal changes in the underlying survival process. This suggests a high probability of the occurrence of phenomena that are considered medical when the initial probability of the outcome is close to zero. Simulation experiments have demonstrated that the distribution differs from the Weibull distribution in estimating the AIC and BIC, the mean, or the least squared error (MSE). The proposed distribution also exhibits the greatest risk in multiple variables over time, indicating a greater risk for those with a shorter or longer duration. Furthermore, the distribution accurately reflects the survival times of cancer patients, as well as definitively, in long-tailed survival times.

References:

[1] H. S. Klakattawi, “Alpha power Kumaraswamy Weibull distribution: Properties and applications to cancer data,” Computational Intelligence and Neuroscience, vol. 2022, pp. 1–14, 2022, doi: 10.1155/2022/5745788.

[2] Y. Li, H. Zhang, J. Wang, and X. Chen, “Application of Weibull parametric survival model in cancer survival analysis,” BMC Medical Research Methodology, vol. 24, p. 87, 2024, doi: 10.1186/s12874-024-02214-7.

[3] H. S. M. Shebib, B. K. Ali, and M. W. Naserallah, “Use the best fitted model for estimating the number of infections with the COVID-19 virus under fuzzy environment,” International Journal of Agricultural and Statistical Sciences, vol. 17, no. 1, pp. 201–206, 2021.

[4] T. N. Sindhu et al., “Distributional properties of the entropy transformed Weibull distribution and applications to various scientific fields,” Scientific Reports, vol. 14, Art. no. 31827, 2024, doi: 10.1038/s41598-024-83132-w.

[5] L. P. Sapkota et al., “New bounded unit Weibull model: Applications with quantile regression,” PLOS ONE, 2025.

[6] A. Z. Afify, R. Alsultan, A. S. Alghamdi, and H. A. Mahran, “A new flexible Weibull distribution for modeling real-life data: Improved estimators, properties, and applications,” AIMS Mathematics, vol. 10, no. 3, pp. 5880–5927, 2025, doi: 10.3934/math.2025270.

[7] A. A. Najm and B. K. Ali, “Survival function estimation for Weibull distribution based on granular hesitant fuzzy set,” Central

Asian Journal of Mathematical Theory and Computer Sciences, vol. 6, no. 1, pp. 26–43, 2025.

[8] D. Alsulami and A. S. Alghamdi, "A new extended Weibull distribution: Estimation methods and applications in engineering, physics, and medicine," Mathematics, vol. 13, no. 20, p. 3262, Oct. 2025, doi: 10.3390/math13203262.

[9] C. G. Papadopoulos and K. Politis, "New approximations for the renewal function of the Weibull distribution," Reliability Engineering & System Safety, vol. 272, pt. 1, p. 112425, Aug. 2026, doi: 10.1016/j.ress.2026.112425.

[10] S. Rangoli, R. Kumar, and A. Patel, "Modified Weibull distribution: Properties and applications in survival data analysis," Cureus, vol. 17, no. 1, p. e326188, 2025, doi: 10.7759/cureus.326188.

[11] H. H. Alsulami, E. M. Almetwally, and R. Alharbi, "A new generalization of the Weibull distribution with statistical properties and applications," Mathematics, vol. 13, no. 20, p. 3262, 2025, doi: 10.3390/math13203262.

[12] Y. Gong, Z. Liu, L. Zhang, and X. Wang, "A transformed inverse Weibull distribution with statistical properties and applications," PLOS ONE, vol. 20, no. 5, p. e0335555, 2025, doi: 10.1371/journal.pone.0335555.

[13] H. S. Klakattawi, "Alpha power Kumaraswamy Weibull distribution: Properties and applications to cancer data," Computational Intelligence and Neuroscience, vol. 2022, pp. 1–14, 2022, doi: 10.1155/2022/5745788.

[14] N. A. Noori, M. A. Khaleel, and R. W. Faris, "Revised neutrosophic new odd Weibull-Weibull distribution with heart pulse count data application," Research Article, ver. 2.

[15] A. Z. Afify, R. Alsultan, A. S. Alghamdi, and H. A. Mahran, "A new flexible Weibull distribution for modeling real-life data: Improved estimators, properties, and applications," AIMS Mathematics, vol. 10, no. 3, pp. 5880–5927, Mar. 2025, doi: 10.3934/math.2025270.