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Different Transformation Methods of the Lomax Distribution: A Review

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ABSTRACT

Over the years, several scholars have attempted to create models for occurrences in which the data distribution exhibits varying degrees of heavy-tailedness. Various generalizations or expansions of the Lomax distribution have been suggested in order to achieve this objective. The behavior of the system is determined by two fundamental parameters: the form shape parameter and the scale parameter. The Lomax distribution effectively represents many failure patterns found in real-world situations. The versatility and wide range of applications make it an essential instrument in the investigation of dependability and survival modeling. This paper provides a comprehensive overview of several techniques for extending Lomax distribution. The included extensions include Lomax distribution, Inverse Transformation, and Alpha Power Transformations. Multiple novel and comparable distributions were deliberated and scrutinized. The research has the potential to serve as a valuable standard and encourage the development of improved distributions that can accurately represent complex events.

1. Introduction

The Lomax model (Lx), which is highly effective for modeling lifetime data and company failure data, was examined by Lomax [1]. The Lx distribution is often referred to as the Pareto Type II distribution. Furthermore, it serves as a crucial method for representing diverse data sets in varied scenarios. The Lx distribution is a probability distribution function (PDF) that has a heavy tail, meaning it has a higher likelihood of extreme values. It is commonly used in business, economic, and actuarial modeling. Various generalizations of the Lx distribution have been created to enhance its flexibility and capability to describe a wider range of exceptional data.

Over the years, numerous academics have examined various generalizations of the Lomax distribution, the beta Lx distribution [2], transmuted Lx distribution [3], exponential Lx distribution [4], The Gumbel-Lx distribution [5], the logistic-Lx distribution [6], Exponentiated Weibull-Lx distribution [7], The half-logistic Lx distribution [8], Logarithm transformed Lx distribution [9], Flexible Lx distribution [10], A generalization of Lx distribution [11], the Topp Leone Kumaraswamy Lx distribution [12], the transmuted generalized Lx distribution [13], The minimum Lindley Lx distribution [14], and On Maxwell–Lx distribution [15]. The Lx distribution has also been developed using inverse Transformation [16]. Zeineldin et al. [17] announced a novel lifetime model called the odd Frechet inverse Lx distribution with its properties and applications, the Topp-Leone Inverse Lx distribution [18], and the

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Kumaraswamy Generalized Inverse Lx distribution [19]. Bulut et al. [20] presented the Alpha Power Transformation, a recent development in Lx distribution

This research aims to review current Lx model developments and examine Inverse Lx and Alpha power transformation approaches. This paper is organized as follows: Section 1 provides an introduction to Lx and its Extension. Section 2 explores the concept of inverse Transformation for the Lx distribution. Section 3 discusses the use of Alpha Power Transformation to Extend the Lx Distribution. Section 4 presents the methodology. Section 5 delivers the conclusion.

2. Some Extension of Lx distribution

The Lx distribution with parameters c and d applies to a random variable X if, for all values of $x > 0$, the cumulative distribution function (cdf) is as follows:

$$G(x) = 1 - \left(1 + \frac{x}{c}\right)^{-d} \tag{1}$$

where d > 0 and c > 0 as shape and scale parameters. The probability density function (pdf) for (1) decreases to

$$g(x) = \frac{d}{c}\left(1 + \frac{x}{c}\right)^{-(d+1)} \tag{2}$$

Where $c, d > 0$.

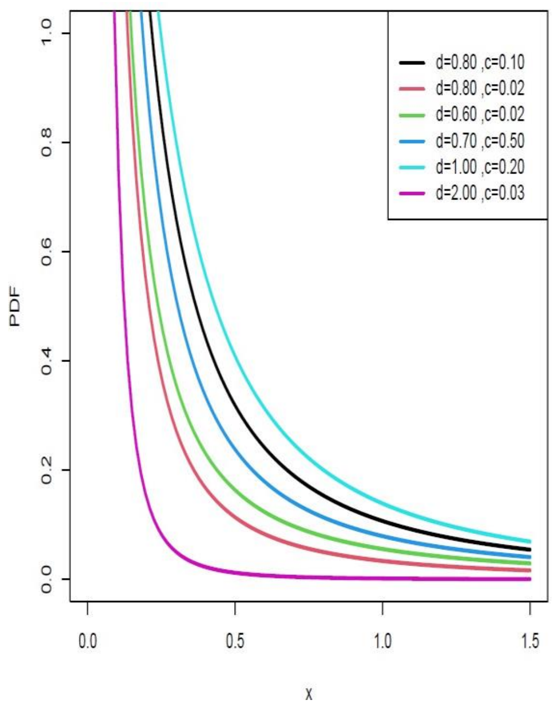

Figure1. Plot pdf for Lx distribution.

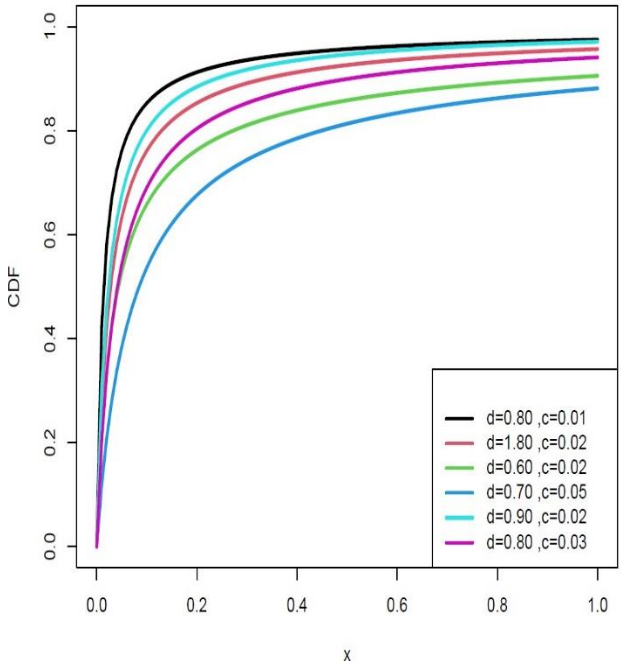

Figure 2. Plot cdf for Lx distribution.

The Gumbel Lomax distribution was proposed by Tahir et al. [5]. The pdf and cdf are defined as:

$$F(x) = e^{-\gamma\left(\left(1+\frac{x}{c}\right)^{-d}-1\right)^{-\frac{1}{\beta}}} \tag{3}$$

and

$$f(x)=\frac{\gamma d\left(1+\frac{x}{c}\right)^{-\frac{d}{\beta}-1}}{c\beta\left(1-\left(1+\frac{x}{c}\right)^{-d}\right)^{\frac{1}{\beta}+1}}e^{-\gamma\left(\left(1+\frac{x}{c}\right)^{-d}-1\right)^{-\frac{1}{\beta}}} \quad (4)$$

where $x,\beta,\gamma,c,d>0$.

Derived statistical features of the model include the ordinary, incomplete moments, generating function, mean deviations, quantile function, order statistic, and Shannon entropy. The model parameters are estimated using maximum likelihood. The performance of the model was evaluated using two datasets and compared models.

Zubair et al. [6] introduced and explored the Logistic Lx distribution (LLxD). The cdf and pdf of the LLxD are specified as:

$$F(x)=\left(1+\left(\log\left(1+\frac{x}{c}\right)^{d}\right)^{-a}\right)^{-1} \quad (5)$$

and

$$f(x)=\frac{ad}{c}\left(1+\frac{x}{c}\right)^{-1}\left(log\left(1+\frac{x}{c}\right)^{d}\right)^{-a-1}$$

$$\times\left(1+\left(\log\left(1+\frac{x}{c}\right)^{d}\right)^{-a}\right)^{-2} \quad (6)$$

Where $x,a,d,c>0$.

The LLxD fits right-skewed and nearly symmetric shape data sets well. Examine its structural aspects, such as a linear density function representation, explicit expressions for ordinary and incomplete moments, generating function, mean deviations, quantile function, and Shannon entropy. Simulation research shows that ML and minimal spacing distance estimators excel in moderate to large samples. Test the LLxD on two real data sets to demonstrate its practicality.

The Exponentiated Weibull Lx distribution (EWLxD) was suggested by Hassan et al. [7]. The cdf and pdf of the new model are as:

$$F(x)=\left(1-e^{\left(-s\left(1+\frac{x}{c}\right)^{d}-1\right)^{\rho}}\right)^{b} \quad (7)$$

and

$$f(x)=\frac{bsd\rho}{c}\left(1+\frac{x}{c}\right)^{d\rho-1}\left(-\rho\left(1+\frac{x}{c}\right)^{d}-1\right)^{\rho-1}$$

$$\times e^{\left(-\rho\left(1+\frac{x}{c}\right)^{d}\right)^{\rho}}\left(1-e^{\left(-\rho\left(1+\frac{x}{c}\right)^{d}\right)^{\rho}}\right)^{b-1} \quad (8)$$

The EWLxD is a variant of the WLx distribution proposed by [5]. Furthermore, it presents a novel model. The EWLxD is analyzed to identify several properties, such as moments, mean residual life, order statistics, quantiles function, renyi, and q-entropies, the paper outlines the procedure for acquiring the ML, least squares, and weighted least squares estimators. Applying the EWLxD to two real data sets has shown that the new model may be successfully used to provide enhanced outcomes.

The Weighted Lx distribution (WLxD) was suggested by Kilany [21]. The cdf and pdf of the WLxD are as:

$$F(x)=\frac{\Gamma(d+1)c^{-k}x^{k}2F_{1}\left(d+1,k,k+1;-\frac{x}{c}\right)}{\mathrm{k}\Gamma(\mathrm{k})\Gamma(1+d-k)} \quad (9)$$

and

$$f(x)=\frac{\Gamma(d+1)c^{1+d-k}}{\Gamma(\mathrm{k})\Gamma(1+d-k)}\left(\frac{x^{k+1}}{(x+c)^{d+1}}\right) \quad (10)$$

Where $2F_1(k,d,c;x)$ is the hypergeometric function and $x\geq 0, d,c>0, 0<k<d+1$.

The aim of this research is to provide a solution or concept. The widespread use of the Lomax model in life testing is the primary driver for the development of the WLxD, which offers more flexibility in the analysis datasets. ML and method of moments techniques estimates are used to compute the parameter estimation of the WLxD, the proposed model, using two data sets, consistently yields a better fit in comparison to the Lomax distribution.

The Flexible Lx distribution (FLxD) was introduced by Ijaz et al. [10]. The FLDs cdf and pdf as:

$$F(x) = 1 - \left(1 + \left(\frac{x}{c}\right)^{a}\right)^{-d} \quad (11)$$

and

$$f(x) = \frac{da}{c^{a}} x^{a-1} \left(1 + \left(\frac{x}{c}\right)^{a}\right)^{-d-1} \quad (12)$$

where $x, a, d, c > 0$.

A new model called the FLx was developed and studied. Features of the data pertaining to statistics were investigated, including the hazard function, survival function, mode, order statistic, and more. ML was used to find to find the FLxD parameters. A study was carried out using simulations to show that the mean square error and bias for the FLxD decrease with increasing size. By applying two real datasets. The FLxD demonstrated superior performance compared to the other distributions that were examined in these lifespan data sets. Therefore, we may deduce that the FLxD is more adaptable than the Lx, Exponential Lx, and POLxO distributions. It is anticipated to exhibit favorable performance when used with suitable data sets.

Bantan et al. [22] introduced the Zubair Lx distribution (ZLD). The cdf and pdf of ZLxD are as:

$$F(x) = \left(e^{\vartheta} - 1\right)^{-1} \left(e^{\vartheta\left(1-\left(1+\frac{x}{c}\right)^{-d}\right)^{2}} - 1\right) \quad (13)$$

and

$$f(x) = 2d\vartheta c^{-1}\left(e^{\vartheta} - 1\right)^{-1} \left(1 + \frac{x}{c}\right)^{-d-1}$$

$$\times \left(1 - \left(1 + \frac{x}{c}\right)^{-d}\right) e^{\vartheta\left(1-\left(1+\frac{x}{c}\right)^{-d}\right)^{2}} \quad (14)$$

where $x, d, \vartheta, c > 0$

A number of model features are investigated and obtained, such as moments, probability-weighted moments, renyi entropy, quantile function, stochastic ordering, mean residual life, and mean waiting time. Both ranked sets and simple random sampling lead to parameter estimators with ML.

The X-gamma Lx distribution (XGLxD) was introduced by Almetwally et al. [23]. The XGLxD cdf and pdf are:

$$F(x) = 1 - \frac{\left(1 + \frac{x}{c}\right)^{-ad}}{a+1}$$

$$\left(1 + a + daln\left(1 + \frac{x}{c}\right) + 0.5a^{2}d^{2}\left(\ln\left(1 + \frac{x}{c}\right)\right)^{2}\right) (15)$$

and

$$f(x) = \frac{ad\left(1 + \frac{x}{c}\right)^{-ad-1}}{c(a+1)}$$

$$\times \left(a + 0.5a^{2}d^{2}\left(\ln\left(1 + \frac{x}{c}\right)\right)^{2}\right) \quad (16)$$

where $x, d, a, c > 0$.

The objective of this study is to propose a solution or idea. The extensive utilization of the Lomax model in life testing is the main catalyst for the development of the XGLxD, which provides more versatility in the analysis of lifetime data. MLE and Maximum a posteriori estimation are employed to calculate the parameter estimation of the XGLxD, while simulation results are used to evaluate the model's performance. The suggested model, utilizing three real-world datasets, consistently provides a superior match compared to the Lx distribution.

Ameeq et al. [24] introduced the Marshall-Olkin Lx distribution (MOLxD). The cdf and pdf of the MOLxD is given by:

$$F(x) = \frac{1-\left(1+\frac{x}{c}\right)^{-d} e^{\left(1+\frac{x}{c}\right)^{-d}}}{1-(1-\theta)\left(\left(1+\frac{x}{c}\right)^{-d} e^{\left(1+\frac{x}{c}\right)^{-d}}\right)^{2}} \quad (17)$$

and

$$f(x) = \frac{\frac{\theta d}{c}\left(1+\frac{x}{c}\right)^{-(2d+1)} e^{\left(1+\frac{x}{c}\right)^{-d}}}{1-(1-\theta)\left(\left(1+\frac{x}{c}\right)^{-d} e^{\left(1+\frac{x}{c}\right)^{-d}}\right)^{2}} \quad (18)$$

Where $x, \theta, d, c > 0$.

The distribution of MOLxD exhibited symmetrical, positively skewed, reversed-J, and inverted bathtub forms, characterized by increasing, decreasing, and alternating patterns of increase and decrease, respectively. Practically, the MOLxD analysis incorporates both failure and insurance data. In addition, a group acceptance sampling plan was constructed using the MOLxD, with the median serving as a quality criterion. We provide numerical and graphical representations of both the ES and VaR. Simulation experiments are performed to assess the proposed estimating approach.

3. Using Inverse Transformation to Extend the Lx distribution

Let X be a random variable with an Inverse Lx distribution (ILxD). If a random variable Z follows a Lx distribution, then $Z = \frac{1}{X}$. Rahman et al. [16] applied the ILx distribution for the first time and investigated ILxD estimation issues with the Bayesian method. For dependability estimates using censored observations. The cdf and pdf of the ILxD is given by:

$$G(x) = \left(1 + \frac{\vartheta}{x}\right)^{-\gamma} \quad (19)$$

and

$$g(x) = \gamma\vartheta x^{-2}\left(1 + \frac{\vartheta}{x}\right)^{-\gamma-1} \quad (20)$$

where $x, \gamma, \vartheta > 0$.

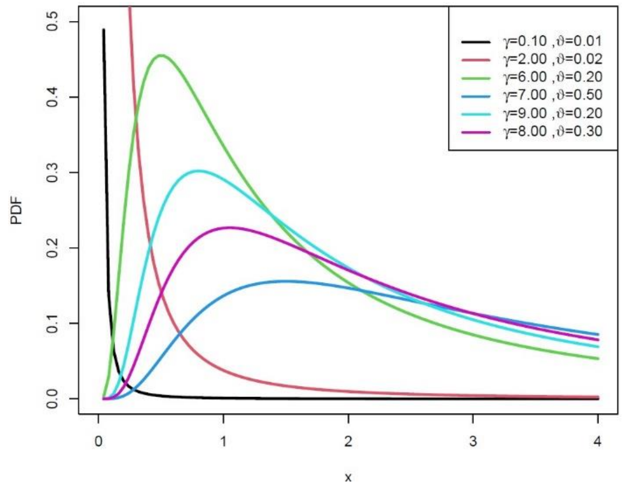

Figure 3. Plot pdf for inverse Lx distribution.

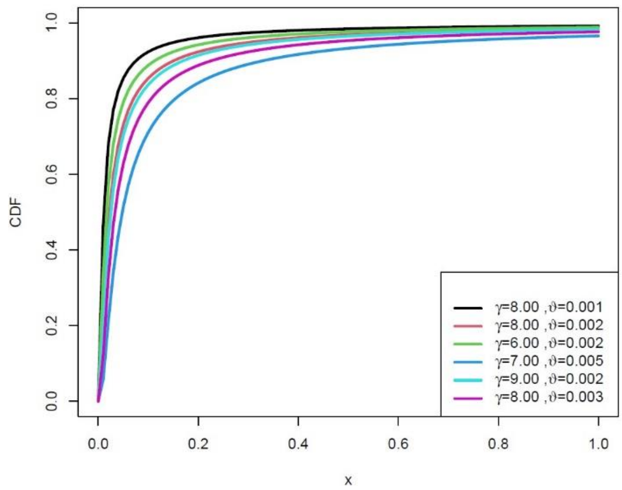

Figure 4. Plot cdf for inverse Lx distribution.

The Marshall Olkin inverse Lx distribution (MOILxD) was introduced by Maxwell et al. [25]. The cdf and pdf of the MOILxD are given:

$$F(x) = \frac{\left(1+\frac{\vartheta}{x}\right)^{-\gamma}}{1-(1-a)\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)} \quad (21)$$

and

$$f(x) = \frac{a\gamma\vartheta\left(1+\frac{\vartheta}{x}\right)^{-\gamma-1}}{x^2\left(1-(1-a)\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)\right)^2} \quad (22)$$

where $x \geq 0, a, \gamma, \vartheta > 0$.

Statistical properties for the MOILxD have been established. The Cancer Stem data set was analyzed using the MOILxD, which demonstrated a superior match compared to the

MO Flexible Weibull Extension Distribution and the Marshall-Olkin exponential Weibull distribution. This conclusion was based on log-likelihood and fit statistics values. Based on our analysis, we can confidently state that the MOILxD is the best suitable model out of the distributions we investigated. It is also a highly competitive model for accurately capturing lifetime phenomena.

The Zubair inverse Lx distribution (ZILxD) was suggested by Falgore [26]. The cdf and pdf of the ZILxD are given:

$$F(x) = \frac{e^{\{c\left(1+\frac{\vartheta}{x}\right)^{-2\gamma}\}-1}}{e^{c}-1} \tag{23}$$

And

$$f(x) = \frac{2c\vartheta\gamma x^{-2}\left(1+\frac{\vartheta}{x}\right)^{-2\gamma-1} e^{\{c\left(1+\frac{\vartheta}{x}\right)^{-2\gamma}\}}}{e^{c}-1} \tag{24}$$

where $x, c, \gamma, \vartheta > 0$.

The ZILxD was introduced and analyzed. The statistical features of the data were examined, including moments, moment generating function, entropy, and order statistics. The parameters of the ZILxD were determined using the maximum likelihood technique. The pdf plots suggest that the form can exhibit right skewness, whereas the hazard function plots describe the shape as being constant, right-skewed, and declining. Furthermore, the cumulative distribution function converges to a value of one. A concrete illustration of actual data sets demonstrates the significance and worth of the novel model.

The Odd Fr´echet Inverse Lx distribution (OFILxD) by ZeinEldin et al. [17]. The cdf and pdf of the OFILxD are given:

$$F(x) = e^{-\left(\left(1+\frac{\vartheta}{x}\right)^{\gamma}-1\right)^{a}} \tag{25}$$

and

$$f(x) = \frac{a\gamma\vartheta}{x^2}\left(1+\frac{\vartheta}{x}\right)^{a\gamma-1}\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)^{a-1} e^{-\left(\left(1+\frac{\vartheta}{x}\right)^{\gamma}-1\right)^{a}} \tag{26}$$

where $x, a, \gamma, \vartheta > 0$.

The introduction and analysis of the OFILxD and detailed. Moments, quantile function, Bowley skewness, Moors kurtosis, ordinary moments, and probability-weighted moments. Were among the statistical aspects of the data that were investigated. The ML, least squares, Pearson correlation, and Anderson Darling methods of estimation parameters. The novelty and value of the concept are highlighted by a practical example using three real datasets.

The Flexible Reduced Logarithmic Inverse Lx distribution (FRLILxD) was introduced by Buzaridah et al. [27]. The cdf and pdf of the FRLILxD are given by:

$$F(x) = 1 - \frac{\log\left(1+c-c\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)}{\log(1+c)} \tag{27}$$

and

$$F(x) = \frac{\frac{c\gamma\vartheta}{x^2}\left(1+\frac{\vartheta}{x}\right)^{-\gamma-1}}{\left(1+c-c\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)\log(1+c)} \tag{28}$$

where $x, c, \gamma, \vartheta > 0$.

The mathematical properties of the FRLILxD include the moments, hazard function, quantile function, and order statistics. The maximum likelihood approach is employed to estimate the model parameters. The empirical evidence has demonstrated that the suggested distribution is highly advantageous for handling dependable data and exhibits superior performance. The image demonstrates that the FRLILxD is the most suitable model for our data sets, in comparison to the other models that were investigated.

The Topp-Leone inverse Lx distribution (TLILxD) was suggested by Soliman and Ismail [17]. The cdf and pdf of the TLILxD is given by:

$$F(x) = \left(1-\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)^{2}\right)^{a} \tag{29}$$

and

$$f(x) = 2a\gamma\vartheta^{-\gamma}x^{\gamma-1}\left(1+\frac{\vartheta}{x}\right)^{-\gamma-1}$$

$$\times\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)\left(1-\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)^{2}\right)^{a-1} \quad (30)$$

where $x, a, \gamma, \vartheta > 0$.

The TLILxD provides well-defined density and distribution function formulae. Its statistical qualities are computed to a certain extent. Given either full or partial data, the reliability estimator is built, and ML estimators of the population parameters are computed. Together with the reliability interval estimator, here provide the estimated confidence intervals for the parameters. The presented estimators are tested in simulation research. The simulation output informs the suggestions. By using two real data sets, the newly developed models' usefulness and importance are shown empirically. Of all the models tested, the TLILxD clearly shows the best agreement with the data.

In their publication [28], Al-Marzouki et al. presented the Half-logistic inverse Lx distribution (HLILxD). What follows is the HLILxD's cdf and pdf:

$$F(x) = \frac{1-\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)^{b}}{1+\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)^{b}} \quad (31)$$

and

$$f(x) = \frac{\frac{2b\gamma\vartheta}{x^2}\left(1+\frac{\vartheta}{x}\right)^{-\gamma-1}\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)^{b-1}}{\left(1+\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)^{b}\right)^{2}} \quad (32)$$

where $x, b, \gamma, \vartheta > 0$.

The aim is to provide a pragmatic extension of the HLILxD that gives further advantages for modeling data with heavy-tailed characteristics. These advantages include more flexibility in both the upper and lower ranges of the prior HLILxD. The statistical characteristics of the proposed distribution, such as first-order stochastic dominance, moments, renyi and q-entropies, and order statistics, are computed. The estimate of parameters is analyzed using six different estimation methodologies. By using the ML technique, three empirical data sets provide evidence of the novel distributions.

By using Truncated method [29]-[35]. Ahmadini et al. [36] introduced the truncated Lomax inverse Lx distribution (TLILxD). The cdf and pdf of the TLILxD is given by:

$$F(x) = (1-2^{-r})^{-1}$$

$$\times\left(1-\left(1+\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)\right)^{-r} \quad (33)$$

and

$$f(x) = \frac{r\gamma\vartheta(1-2^{-r})^{-1}}{x^2}\left(1+\frac{\vartheta}{x}\right)^{-\gamma-1}$$

$$\times\left(1-\left(1+\left(1+\frac{\vartheta}{x}\right)^{-\gamma}\right)\right)^{-r-1} \quad (34)$$

where $x, r, \gamma, \vartheta > 0$.

The TLILxD is a flexible way to describe lifespan data. The equations for density and distribution functions are obtained by linearly mixing the inverse Lomax distribution. The new model undergoes analysis to ascertain many mathematical statistics, such as probability-weighed moments, quantile function, moments generating function, ordinary and incomplete moments, inverse moments, conditional moments, and renyi entropy. The ML method is used to calculate the point estimate and estimated confidence interval of population parameters. We assess the accuracy of estimations using a simulation study. The relevance and usefulness of the novel model are shown by its application to two real datasets, in contrast to many alternative models.

The Kumaraswamy Generalized Inverse Lx distribution (KGILxD) was suggested by Ogunde et al. [37]. The cdf and pdf of the KGILxD is given by:

$$F(x) = \left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma a}\right)^{c} \quad (35)$$

and

$$f(x) = ca\gamma\vartheta x^{-2}\left(1+\frac{\vartheta}{x}\right)^{-(\gamma a+1)}$$
$$\times\left(1-\left(1+\frac{\vartheta}{x}\right)^{-\gamma a}\right)^{c-1} \qquad (36)$$

where $x, c, a, \gamma, \vartheta > 0$.

The reliability function and model parameters' approximate confidence intervals are determined. The validity of the parameter estimate approach of the KGILxD is shown via a simulation exercise. The new model is shown to be more applicable using three actual data sets. The data utilized in this research demonstrate that the KGILxD outperforms all existing lifespan models.

4. Using Alpha Power Transformation to Extend the Lx distribution

Mahdavi and Kundu [38] developed the cdf of the Alpha power transformation family of distribution and This method has been employed by numerous researchers to produce flexible distributions, as demonstrated in publications [39]-[50]. Based on a specified cdf and pdf of a baseline distribution.

$$F(x) = \begin{cases} \frac{\alpha^{G(x)}-1}{\alpha-1} & ; if\ \alpha > 0, \alpha \neq 1 \\ G(x) & ; if\ \alpha = 1 \end{cases} \qquad (37)$$

and

$$f(x) = \begin{cases} \frac{log\alpha}{\alpha-1}\alpha^{G(x)}f(x) & ; if\ \alpha > 0, \alpha \neq 1 \\ f(x) & ; if\ \alpha = 1 \end{cases} \qquad (38)$$

The Alpha-Power Transformed Lx distribution (APTLxD) was introduced by Amer [51]. The cdf and pdf of the APTLxD are provided as follows:

$$F(x) = \begin{cases} \frac{1}{\alpha-1}\left(\alpha\alpha^{-\left(1+\frac{x}{\vartheta}\right)^{-\gamma}}-1\right) & ; if\ \alpha,\gamma,\vartheta > 0, \alpha \neq 1 \\ 1-\left(1+\frac{x}{\vartheta}\right)^{-\gamma} & ; if\ \gamma,\vartheta > 0, \alpha = 1 \end{cases} \qquad (39)$$

and

$$f(x) = \begin{cases} \frac{\alpha\gamma log\alpha}{\alpha-1}\left(1+\frac{x}{\vartheta}\right)^{-\gamma-1}\alpha^{-\left(1+\frac{x}{\vartheta}\right)^{-\gamma}} & ; if\ \alpha,\gamma,\vartheta > 0, \alpha \neq 1 \\ \frac{\gamma}{\vartheta}\left(1+\frac{x}{\vartheta}\right)^{-\gamma-1} & ; if \gamma,\vartheta > 0, \alpha = 1 \end{cases} \qquad (40)$$

The APTLxD has been analyzed and several structural aspects have been proposed and calculated. These include moments, generating function, quantiles, reliability function, Bonferroni and Lorenz curves, renyi and Shannon entropy, and order statistics. Furthermore, the ML technique is used to estimate the parameters of the model. Simulation techniques are created to get ML algorithms to result in decreased bias and root mean square error.

The Alpha power Lx distribution (APLxD) was suggested by Bulut et al. [20]. The cdf and pdf of the APLxD are provided as follows:

$$F(x) = \begin{cases} \frac{\alpha^{1-\left(1+\frac{x}{\vartheta}\right)^{-\gamma}}-1}{\alpha-1} & ; if\ \alpha,\gamma,\vartheta > 0, \alpha \neq 1 \\ 1-\left(1+\frac{x}{\vartheta}\right)^{-\gamma} & ; if\ \gamma,\vartheta > 0, \alpha = 1 \end{cases} \qquad (41)$$

and

$$f(x) = \begin{cases} \frac{log\alpha\gamma}{\vartheta(\alpha-1)}\left(1+\frac{x}{\vartheta}\right)^{-\gamma-1}\alpha^{1-\left(1+\frac{x}{\vartheta}\right)^{-\gamma}} & ; if\ \alpha,\gamma,\vartheta > 0, \alpha \neq 1 \\ \frac{\gamma}{\vartheta}\left(1+\frac{x}{\vartheta}\right)^{-\gamma-1} & ; if \gamma,\vartheta > 0, \alpha = 1 \end{cases} \qquad (42)$$

The Gull Alpha Power Lx distribution (GAPLxD) was introduced by Tolba et al. [52]. The cdf and pdf of the GAPLxD are shown below:

$$F(x) = \frac{\alpha\left(1-\left(1+\frac{x}{\vartheta}\right)^{-\gamma}\right)}{\alpha^{1-\left(1+\frac{x}{\vartheta}\right)^{-\gamma}}} \quad , \alpha,\gamma,\vartheta > 0, \alpha \neq 1 \qquad (43)$$

and

$$f(x) = \frac{\gamma}{\vartheta}\left(1+\frac{x}{\vartheta}\right)^{-\gamma-1}\alpha^{\left(1+\frac{x}{\vartheta}\right)^{-\gamma}}$$
$$\times\left(1-log\alpha\left(1-\left(1+\frac{x}{\vartheta}\right)^{-\gamma}\right)\right) \qquad (44)$$

where $\alpha, \gamma, \vartheta > 0, \alpha \neq 1$

Quantile function, Moments, order statistics, mean residual life, renyi entropy, skewness, kurtosis so on are all proposed or deduced structural aspects of the GAPLxD. The development of simulation has led to a decrease in bias and mean square error with increasing sample sizes. In comparison to other models derived using critical information and measure statistics, the GAPLxD shows more strength.

5. Methodology

5.1Mathematical Properties

Important mathematical features for the Lx distribution in discussion are:

a) Moments
b) Incomplete moments
c) probability weighted moments
d) inverse moments
e) Quantile function
f) mean deviations
g) Reliability function
h) Bonferroni curve
i) Lorenz curve
j) Rényi entropy
k) q-entropies
l) Shannon entropy
m) Order statistics

5.2 Parameter Estimation

a) maximum likelihood
b) least squares
c) Pearson correlation
d) Anderson-Darling

5.3 Modeling of Data

Three modeling phases are used to determine whether a data collection can be modeled using one or more extensions of Lomax models:

Step 1: Model Selection

Step 2: Estimation of model parameters

Step 3: Goodness of fit test

6. Conclusion

The article explores several expansions of the Lomax model and examines their distinct properties and form behaviors. Our research found that these extensions of Lomax models are more suited for modeling complicated data sets due to the diverse range of forms in the reliability and hazard rate function. Next, we will discuss the techniques used for estimating parameters, the mathematical characteristics, and the process of modeling data. Lately, a novel approach for transformation called the Alpha Power Transformation (APL) was presented by [38] and a new method for inverse transformation was used by [16]. The simulated behavior of several Lomax distribution extensions was analyzed. It was shown that the bias and mean square error for different parameter values decrease with increasing sample size n. This indicates that IL and APT techniques are reliable on the Lomax distribution.

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