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Sine Type II Topp-Leone Burr XII Distribution: Mathematical Properties and Application to Biomedical and Environmental Data

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ABSTRACT

This paper introduces a novel three-parameter lifetime model known as the Sine Type II Top-Leone Burr XII distribution. We derive various mathematical properties of this distribution, including its survival function, hazard rate function, quantile and generating functions, moments, Rényi entropy, moment generating function, and order statistics. The parameters of the proposed model were estimated using the maximum likelihood method. A simulation study examines the consistency of these estimates across both small and large sample sizes. To demonstrate the flexibility and significance of the distribution, we apply it to model Biomedical and Environmental data sets.

1. Introduction

Understanding and accurately modelling real-life data is crucial for advancing knowledge across diverse fields such as finance, engineering, environmental science, biomedical sciences, and geophysics. A pivotal aspect of this endeavour is the selection of an appropriate statistical distribution, which directly influences the quality of statistical analysis. Despite the development and application of numerous extended distributions to model survival data, significant challenges persist, particularly when dealing with complex real-world data.

In this study, we focus on the Burr XII (BXII) distribution, originally proposed by [1]. This distribution has proven versatile, with numerous applications including failure time modelling, reliability analysis, and use in engineering and medical science. For instance, [2] utilized the three-parameter BXII distribution to model extreme events like flood frequency, while [3] explored the BXII model and its related distributions such as the log-logistic, compound Weibull gamma, Pareto II (Lomax), and Weibull exponential distributions.

Motivated by the extensive use and proven utility of the BXII distribution, several generalizations have been proposed by [3] to [12] to enhance its flexibility. These generalizations were obtained using different types of distribution families. Notably, the Sine

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Type II Topp-Leone family, a recent innovation by [13], incorporates a shape parameter and a trigonometric function, offering a versatile tool for enhancing traditional distributions.

In this paper, we propose a novel combination of the Sine Type II Topp-Leone distribution with the BXII distribution, aiming to create a more robust and flexible model. This new distribution is designed to address unresolved challenges in modelling real data, providing a powerful tool for statistical analysis across various fields. Our objective is to extend the Burr XII distribution's applicability, thereby contributing to the advancement of statistical modelling techniques and their practical applications.

2. Methodology

The cumulative distribution function (CDF) of the Sine Type II Topp-Leone generator as defined by [14] is given by:

$$\mathrm{F(x)} = \sin\left\{\frac{\pi}{2}\left[1-(1-(\mathrm{G(x)})^2)^{\alpha}\right]\right\}. \tag{1}$$

and the corresponding probability density function (PDF) is given as

$$f(x) = \frac{\pi}{2} 2\alpha g(x) G(x)\left[1-(G(x))^2\right]^{\alpha-1} \cos\left\{\frac{\pi}{2}\left[1-\left(1-(G(x))^2\right)^{\alpha}\right]\right\}. \tag{2}$$

where θ is the shape parameter, g(x) and G(x) are the PDF and CDF of any baseline distribution. And the CDF and PDF of the baseline (Burr XII) distribution are given in (3) and (4) below:

$$G(x) = 1-\frac{1}{(1+x^{\alpha})^{\beta}}, x > 0 \tag{3}$$

$$g(x) = \frac{\alpha\beta x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}}, \tag{4}$$

Where $\alpha > 0$ and $\beta > 0$ are shape parameters. Substituting (3) and (4) into (1) and (2), we have the CDF of the Sine Type II Topp-Leone Burr XII (STIITLBXII) distribution as follows:

$$\mathrm{F(x)} = \sin\left\{\frac{\pi}{2}\left[1-\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^2\right)^{\theta}\right]\right\}, x\ \alpha, \theta > 0 \tag{5}$$

and the corresponding PDF is given by:

$$f(x) = \frac{\pi}{2}\frac{2\alpha\beta\theta x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}}\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^2\right]^{\theta-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^2\right)^{\theta}\right]\right\} \tag{6}$$

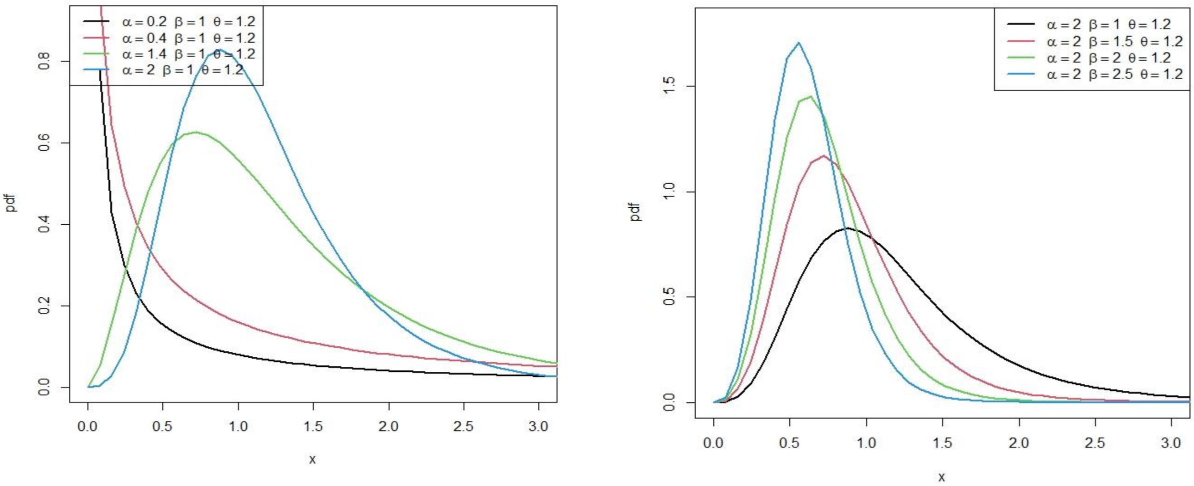

Figure 1: pdf plots of STIITL-BXII distribution for varying parameter values of α, β and θ

The survival function S(x), hazard function h(x), reverse hazard function r(x) and the quantile function Q(u) are given by equation (7) to equation (10).

$$S(x)=1-\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right]^{\alpha}\right]\right\}. \tag{7}$$

$$h(x)=\frac{\frac{\pi}{2}\frac{2\alpha\beta\theta x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}}\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right]^{\alpha-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\alpha}\right]\right\}}{1-\sin\left\{\frac{\pi}{2}\left[1-\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right]^{\alpha}\right]\right\}}. \tag{8}$$

$$r(x)=\frac{\pi}{2}\frac{2\alpha\beta\theta x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}}\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right]^{\alpha-1}\cot\left\{\frac{\pi}{2}\left[1-\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\alpha}\right]\right\}. \tag{9}$$

$$Q(u)=\left(-\frac{2}{\theta}\ln\left(1-\left(1-\left(1-\left(\frac{\sin^{-1}(u)}{\pi/2}\right)\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{2}}. \tag{10}$$

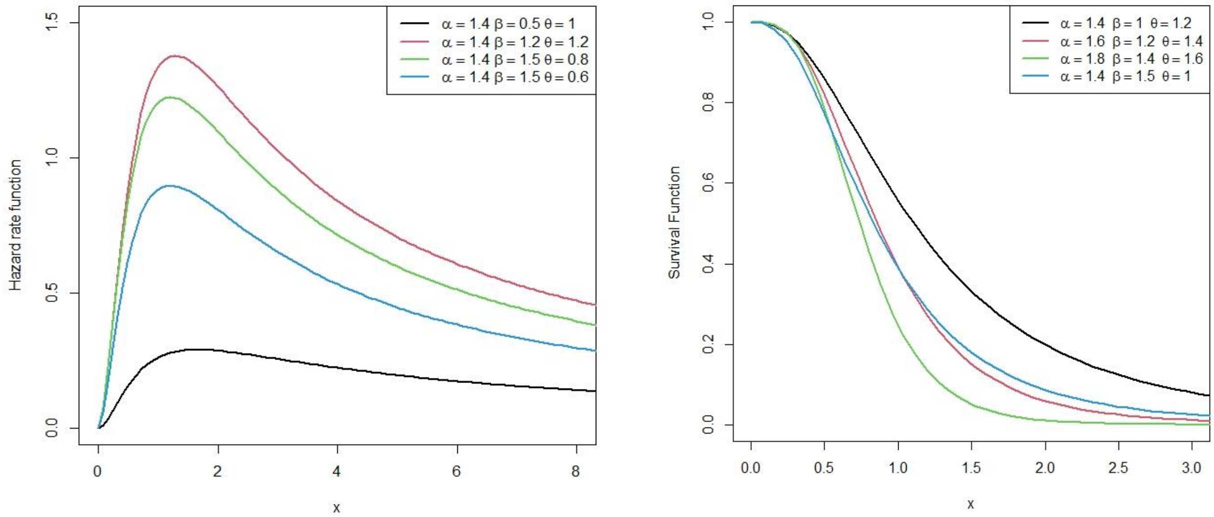

Figure 2: Hazard and Survival plots of STIITL-BXII distribution

3. Some Statistical Properties

3.1 Linear Expansion of the PDF

The PDF and the CDF of the Sine Type II Topp-Leone Rayleigh distribution in equation (5) and (6) can be expanded using power series expansion as follows:

The Taylor's series expansion of Sine and Cosine functions can be expressed as follows:

$$\cos(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{2i!}, \tag{11}$$

$$\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}. \tag{12}$$

Then, employing expansion (10) in the last term of PDF in equation (6) follows:

$$\cos\left\{\frac{\pi}{2}\left[1-\left(1-\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)^2\right]^\alpha\right]\right\} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\frac{\pi^{2i}}{2^{2i}}\left[1-\left(1-\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)^2\right)^\alpha\right]^{2i}. \tag{13}$$

Again, using binomial expansion in (13) gives

$$f(x) = \frac{\pi}{2}\frac{2\alpha\beta\theta x^{\alpha-1}}{(1+x^\alpha)^{\beta+1}}\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)\left[1-\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)^2\right]^{\alpha-1}$$

$$\times\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\frac{\pi^{2i}}{2^{2i}}\left[1-\left(1-\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)^2\right)^\alpha\right]^{2i}$$

Also, $$\left[1-\left(1-\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)^2\right)^\alpha\right]^{2i} = \sum_{j=0}^{\infty}(-1)^j\binom{2i}{j}\left(1-\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)^2\right)^{\alpha j}, \tag{14}$$

Then, inserting equation Therefore, equation (6) becomes;

$$f(x) = \frac{\pi}{2}\frac{2\alpha\beta\theta x^{\alpha-1}}{(1+x^\alpha)^{\beta+1}}\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)\left[1-\left(1-\frac{1}{(1+x^\alpha)^\beta}\right)^2\right]^{\alpha-1}$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j}}{(2i)!}\frac{\pi^{2i}}{2^{2i}}\binom{2i}{j}\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\alpha j} \quad (15)$$

Again, using binomial expansion,

$$\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\alpha j+\alpha-1}=\sum_{k=0}^{\infty}(-1)^{k}\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2k}.$$

(16)

$\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}$ can further be simplified using binomial series expansion as follows:

$$\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2k+1}=\sum_{l=0}^{2k+1}(-1)^{l}\binom{2k+1}{l}(1+x^{\alpha})^{-\beta l},$$

$$f(x)=\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{l=0}^{2k+1}\frac{(-1)^{i+j+k+l}}{(2i)!}\frac{\pi^{2i+1}}{2^{2i}}\binom{2i}{j}\binom{\alpha j+\alpha-1}{k}\binom{2k+1}{l}\frac{2\alpha\beta\theta x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}},$$

Therefore, the PDF can be expressed as:

$$f(x)=\sum_{i,k=0}^{\infty}\Phi_{i,j,k,l}\,\alpha\theta x^{\alpha-1}(1+x^{\alpha})^{-(\beta l+\beta+1)}, \quad (17)$$

Where

$$\Phi_{i,j,k,l}=\sum_{j=0}^{2i}\sum_{l=0}^{2k+1}\frac{(-1)^{i+j+k+l}}{(2i)!}\frac{\pi^{2i+1}}{2^{2i}}\binom{2i}{j}\binom{\alpha j+\alpha-1}{k}\binom{2k+1}{l}.$$

The cdf of the TIITLR distribution in equation (5) can be expanded as follows:

$$\sin\left(\frac{\pi}{2}\left[1-\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\theta}\right]\right)=\sum_{n=0}^{\infty}\frac{(-1)^{n}\pi^{2n+1}}{(2n+1)!2^{2n+1}}\left[1-\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\theta}\right]^{2n+1},$$

Also,

$$\left[1-\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\theta}\right]^{2n+1}=\sum_{m=0}^{2n+1}(-1)^{m}\binom{2n+1}{m}\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\alpha m},$$

New, equation (5) becomes;

$$F(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{2n+1}\frac{(-1)^{n+m}\pi^{2n+1}}{(2n+1)!2^{2n+1}}\binom{2n+1}{m}\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\theta m},$$

Expanding $\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\theta m}$ using binomial expansion,

$$\left(1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2}\right)^{\theta m}=\sum_{p=0}^{\infty}(-1)^{p}\binom{\alpha m}{p}\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2p},$$

Equation (5) can be written as;

$$F(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{2n+1}\sum_{p=0}^{\infty}\frac{(-1)^{n+m}\pi^{2n+1}}{(2n+1)!2^{2n+1}}\binom{2n+1}{m}\binom{\alpha m}{p}\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2p},$$

$\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2p}$ can further be expanded using binomial expansion as follows;

$$\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{2p} = \sum_{q=0}^{2p}(-1)^{q}\binom{2p}{q}(1+x^{\alpha})^{-\beta q},$$

Therefore the cdf in equation (5) becomes;

$$F(x) = \sum_{n,m,p,q=0}^{\infty}\Psi_{m,n,p,q}(1+x^{\alpha})^{-\beta q}. \tag{18}$$

where

$$\Psi_{m,n,p,q} = \sum_{m=0}^{2n}\sum_{q=0}^{2p}\frac{(-1)^{n+m+p+1}}{(2n+1)!}\frac{\pi^{2n+1}}{2^{2n+1}}\binom{2n+1}{m}\binom{\alpha m}{p}\binom{2p}{q}.$$

3.2 Moment

Let X be a random sample from STIITL-BXII distribution, the r^{th} moment of X is obtained as follow:

$$\mu_r = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx, \tag{19}$$

inserting the PDF in (17) gives;

$$\mu_r = \sum_{i,k=0}^{\infty}\Phi_{i,j,k,l}\alpha\theta\beta\int_0^{\infty} x^{r+\alpha-1}(1+x^{\alpha})^{-(\beta l+\beta+1)}dx, \tag{20}$$

By integrating with respect to x, equation (20) can be expressed as:

$$\mu_r = \sum_{i,k=0}^{\infty}\Phi_{i,j,k,l}\theta\beta\frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma\left(\beta q+\beta-\frac{r}{\alpha}\right)}{\Gamma\left(\beta q+\beta-1\right)}, \tag{21}$$

Therefore, the first, second, third and forth moments can be obtained by substituting $r = 1, 2, 3$ and 4 respectively.

$$\mu_1 = \sum_{i,k=0}^{\infty}\Phi_{i,j,k,l}\theta\beta\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma\left(\beta q+\beta-\frac{1}{\alpha}\right)}{\Gamma\left(\beta q+\beta-1\right)}$$

$$\mu_2 = \sum_{i,k=0}^{\infty}\Phi_{i,j,k,l}\theta\beta\frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma\left(\beta q+\beta-\frac{2}{\alpha}\right)}{\Gamma\left(\beta q+\beta-1\right)}$$

$$\mu_3 = \sum_{i,k=0}^{\infty}\Phi_{i,j,k,l}\theta\beta\frac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma\left(\beta q+\beta-\frac{3}{\alpha}\right)}{\Gamma\left(\beta q+\beta-1\right)}$$

$$\mu_4 = \sum_{i,k=0}^{\infty} \Phi_{i,j,k,l} \theta\beta \frac{\Gamma\left(\frac{4}{\alpha}+1\right)\Gamma\left(\beta q+\beta-\frac{4}{\alpha}\right)}{\Gamma\left(\beta q+\beta-1\right)}$$

3.3 Renyi Entropy

Let X be an arbitrary variable with parameter θ that follows the STIITL-BXII distribution. The Renyi entropy of X with $\varphi = (\theta)$ by is given as:

$$I_\lambda(x) = \frac{1}{1-\lambda}\int_{-\infty}^{\infty} f(x)^\lambda \, dx . \tag{22}$$

For the Sine Type II Topp-Leone Burr XII distribution is given by:

$$f(x)^\lambda = \left(\sum_{i,j,k,l=0}^{\infty} \Phi_{i,j,k,l}\alpha\beta\theta x^{\alpha-1}(1+x^\alpha)^{-(\beta l+\beta+1)}\right)^\lambda ,$$

$$f(x;\theta)^\lambda = \left(\sum_{i,j,k,l=0}^{\infty} \Phi_{i,j,k,l}\alpha\beta\theta\right)^\lambda \left(x^{\alpha-1}(1+x^\alpha)^{-(\beta l+\beta+1)}\right)^\lambda ,$$

$$I_\delta(x,\theta) = \frac{1}{1-\lambda}\left(\sum_{i,j,k,l=0}^{\infty} \Phi_{i,j,k,l}\alpha\beta\theta\right)^\lambda \int_0^\infty x^{\lambda(\alpha-1)}(1+x^\alpha)^{-\lambda(\beta l+\beta+1)} \, dx, \tag{23}$$

By integrating the last part equation (23), the Renyi Entropy is given by:

$$I_\lambda(x) = \frac{1}{(1-\lambda)}\left(\sum_{i,j,k,l=0}^{\infty} \Phi_{i,j,k,l}\beta\theta\right)^\lambda \frac{\Gamma(\lambda)\Gamma\left(\alpha\beta\lambda(l+1)+\lambda r+\lambda-1\right)}{\Gamma\left(\lambda(\beta l+\beta+1)\right)} \tag{24}$$

Where $\lambda \neq 1, and\ \lambda > 0$

4.0 Parameter Estimation

4.1 Maximum Likelihood Method

The maximum likelihood method was employed to estimate the parameters of the proposed STIITL-BXII distribution. The likelihood function of the PDF in equation (6) is given by:

$$l = n\log\left(\frac{\pi}{2}\right) + n\log 2 + n\log\alpha + n\log\beta + n\log\theta + (\alpha-1)\sum_{i=1}^{n}\log x_i - (\beta+1)\sum_{i=1}^{n}\left(1+x^\alpha\right) + \sum_{i=1}^{n}\left(1-\frac{1}{\left(1+x^\alpha\right)^\beta}\right)$$

$$+(\theta-1)\sum_{i=1}^{n}\log\left[1-\left(1-\frac{1}{\left(1+x^\alpha\right)^\beta}\right)^2\right] + \sum_{i=1}^{n}\log\left(\cos\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-\frac{1}{\left(1+x^\alpha\right)^\beta}\right)^2\right]^\theta\right]\right\}\right) . \tag{25}$$

Differentiating equation (25) partially with respect to α yields the following expressing.

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log x-(\beta-1)\sum_{i=1}^{n}\log\frac{x^{\alpha}\log x}{(1+x^{\alpha})}+\sum_{i=1}^{n}\frac{x^{\alpha}\log x}{(1+x^{\alpha})}\left(1-\frac{1}{\left(1+x^{\alpha}\right)^{\beta}}\right)^{-1}+2(\theta-1)\sum_{i=1}^{n}\frac{x^{\alpha}\log x}{(1+x^{\alpha})}$$
$$\left(1-\frac{1}{\left(1+x^{\alpha}\right)^{\beta}}\right)\left[1-\left(1-\frac{1}{\left(1+x^{\alpha}\right)^{\beta}}\right)^{2}\right]^{-1}\tan\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-1-\frac{1}{\left(1+x^{\alpha}\right)^{\beta}}\right)^{2}\right]^{\alpha}\right]\right\}. \qquad (26)$$

Differentiating equation (25) partially with respect to θ will gives us the following expressing.

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}-\frac{1}{2}\sum_{i=1}^{n}x_i^{2}+\sum_{i=1}^{n}\frac{x_i^{2}e^{-\left(\frac{\theta}{2}x_i^{2}\right)}}{2\left(1-e^{-\left(\frac{\theta}{2}x_i^{2}\right)}\right)}-(\alpha-1)\sum_{i=1}^{n}x_i^{2}e^{-\left(\frac{\theta}{2}x_i^{2}\right)}\left(1-e^{-\left(\frac{\theta}{2}x_i^{2}\right)}\right)\left[1-\left(1-e^{-\left(\frac{\theta}{2}x_i^{2}\right)}\right)^{2}\right] \qquad (27)$$
$$-\sum_{i=1}^{n}\frac{\pi}{2}\alpha x_i^{2}e^{-\left(\frac{\theta}{2}x_i^{2}\right)}\left(1-e^{-\left(\frac{\theta}{2}x_i^{2}\right)}\right)\left[1-\left(1-e^{-\left(\frac{\theta}{2}x_i^{2}\right)}\right)^{2}\right]^{\alpha-1}\tan\left\{\frac{\pi}{2}\left[1-\left(1-e^{-\left(\frac{\theta}{2}x_i^{2}\right)}\right)^{2}\right]^{\alpha}\right\}.$$

Equations (26) and (27) are the maximum likelihood estimates of the parameters α and θ respectively.

4.2 Assessing the Consistency of the Parameter Estimates

To evaluate the effectiveness of the proposed Sine-Type II Tope Leone-Burr XII distribution, simulation was carried out using the Monte Carlo Simulation method. This aimed to determine the accuracy of parameter estimates derived from maximum likelihood estimates, by computing their mean, bias, and root mean square error. Simulated datasets were generated using the quantile function outlined in equation (10), varying sample sizes from n = 20, 50, 70, 100, 150, 200, 300, 500 and 1000, each replicated 1000 times. Parameter combinations $\alpha=0.5, \beta=5.0$ and $\theta=1.0$ and $\alpha=0.1, \beta=3.3$ and $\theta=1.1$ were tested for each sample size. The estimation results, along with bias and root mean square error, are summarized in Table 1.

Table 4.1: Simulation of Sine Type II Topp-Leone Burr XII distribution

n	Properties	$\alpha=0.1$	$\beta=5$	$\theta=1$	$\alpha=0.1$	$\beta=3.3$	$\theta=1.1$
20	Est.	0.6146	5.5305	0.0486	0.5340	3.3175	0.0631
	Bias	0.5146	0.5305	-0.9514	0.4340	0.0175	-1.0369
	RMSE	0.5485	0.6545	0.9519	0.4737	0.6014	1.0376
50	Est.	0.5932	5.6089	0.0490	0.5128	3.2778	0.0653
	Bias	0.4932	0.6089	-0.9510	0.4128	-0.0222	-1.0347
	RMSE	0.5080	0.6539	0.9515	0.4487	0.5669	1.0353
70	Est.	0.5880	5.6285	0.0491	0.4947	3.2756	0.0665
	Bias	0.4880	0.6285	-0.9509	0.3947	-0.0244	-1.0335
	RMSE	0.4995	0.6535	0.9514	0.4285	0.5517	1.0341
100	Est.	0.5705	5.6366	0.0510	0.4859	3.2492	0.0680
	Bias	0.4705	0.6366	-0.9490	0.3859	-0.0508	-1.0320
	RMSE	0.4807	0.6554	0.9495	0.4173	0.5208	1.0326
150	Est.	0.5399	5.6280	0.0533	0.4760	3.2427	0.0688
	Bias	0.4399	0.6280	-0.9467	0.3760	-0.0573	-1.0312
	RMSE	0.4492	0.6486	0.9472	0.4044	0.4911	1.0318

200	Est.	0.5210	5.6165	0.0547	0.4619	3.2476	0.0699
	Bias	0.4210	0.6165	-0.9453	0.3619	-0.0524	-1.0301
	RMSE	0.4297	0.6381	0.9458	0.3880	0.4864	1.0307
300	Est.	0.4939	5.6135	0.0575	0.4634	3.2065	0.0706
	Bias	0.3939	0.6135	-0.9425	0.3634	-0.0935	-1.0294
	RMSE	0.4013	0.6406	0.9431	0.3885	0.4725	1.0300
500	Est.	0.4613	5.6130	0.0609	0.4416	3.2066	0.0738
	Bias	0.3613	0.6130	-0.9391	0.3416	-0.0934	-1.0262
	RMSE	0.3676	0.6517	0.9397	0.3660	0.4753	1.0268
1000	Est.	0.4157	5.5906	0.0659	0.4170	3.1885	0.0772
	Bias	0.3157	0.5906	-0.9341	0.3170	-0.1115	-1.0228
	RMSE	0.3206	0.6718	0.9346	0.3399	0.4962	1.0234

Table 2 presents a simulation study of the Sine Type II Topp-Leone Burr XII distribution, showing the estimated parameters, bias, and root mean square error (RMSE) for different sample sizes (n) and parameter values α, β and θ. The results are reported for two sets of parameter combinations: $\alpha = 0.1$, $\beta = 5$, $\theta = 1$ and $\alpha = 0.1$, $\beta = 3.3$, $\theta = 1.1$. As the sample size increases from 20 to 1000, the estimated values for each parameter generally get closer to the true parameter values, indicating improved estimation accuracy. Bias and RMSE decrease with larger sample sizes, showing more precise and reliable estimates.

5. Application

The first data set is the relief times of twenty patients receiving an analgesic is by [15]. The data sets are as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The second dataset has thirty consecutive values of precipitation (in inches) in the month of March in Minneapolis, as provided by [16] and recently used by [17]. The data are as follows:

0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05

Table 3: MLE, AIC, BIC, CAIC and HQIC of the two datasets

MODEL	α	β	θ	λ	-LL	AIC	CAIC	BIC	HQIC
STIITLBXII	8.5559	0.0620	6.4606	-	-15.8815	37.7631	39.2601	40.7503	38.3462
TLEBXII	6.7344	0.3255	2.1806	1.9714	-15.4249	38.8499	41.5166	42.8329	39.6274
SBXII	13.048	0.0013	-	-	-32.6757	69.3115	70.0574	71.3429	69.7403
BXII	8.1698	2.6948	-	-	-31.7233	67.447	68.1525	69.4381	67.8354

MODEL	α	β	θ	λ	-LL	AIC	CAIC	BIC	HQIC
STIITLBXII	1.6053	0.2743	6.4751	-	-38.510	82.8676	84.4676	85.5848	83.4507
TLEBXII	1.4798	0.0302	0.1416	13.353	-78.639	151.277	150.354	147.974	149.933
SBXII	8.8319	0.0013	-	-	-46.281	96.5617	97.0061	99.3641	97.4582
BXII	5.0722	14.938	-	-	-44.371	92.7429	93.1873	95.5453	93.6394

Based on the model selection criteria, that is, the AIC, CAIC, BIC, and HQIC, the STIITLBXII model is the best fit for both datasets, as it consistently has the lowest values across all indices, indicating a superior balance between model complexity and fit to the data.

This suggests that STIITLBXII provides the most efficient representation with minimal information loss.

6.0 Conclusion

The application of the STIITLBXII model to two distinct datasets: one involving relief times for patients receiving analgesics and the other consisting of precipitation data in Minneapolis demonstrates the superior performance of the proposed model across various model selection criteria, including AIC, CAIC, BIC, and HQIC. For both datasets, the STIITLBXII model achieved the lowest values for these criteria, indicating it is the most efficient model in fitting the two datasets. Furthermore, the Monte Carlo simulation studies validate the consistency and accuracy of the parameter estimates of the STIITLBXII distribution across various sample sizes, demonstrating that as the sample size increases, the parameter estimates become more accurate, and both bias and RMSE decrease.

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