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Improved Estimator for Population Variance using two Auxiliary Information

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ABSTRACT

This paper introduces a novel exponential estimator employing two auxiliary variables to estimate an unknown population variance. The estimator is designed to improve efficiency as compared with existing estimators. The bias and mean squared error (MSE) of the proposed estimator are obtained by a first-order Taylor series expansion. A theoretical comparison with several existing estimators is presented to demonstrate its superiority. To validate the theoretical results, an empirical study is conducted using three population datasets, complemented by a simulation study for cross-validation. The findings indicate that the proposed estimator consistently provides more precise estimates than the existing estimators. Additionally, we introduced a hybrid approach combining the exponential-type estimator with regression adjustment, further reducing bias and MSE under correlated auxiliary information.

1. Introduction

Estimating population variance is a fundamental problem in survey sampling, as it provides essential information for inference, resource allocation, and policy decisions. In agriculture and food industries, researchers are often interested in the variation in crop yields or nutrient content. To study this variation in the study variable of interest (say, crop yields y), the problem of estimating the population variance S_y^2 of y has also received much attention in survey sampling.It is well known that the use of auxiliary information at the estimation stage improves the estimates of the population parameters of the study variable y. For this case, it is assumed that the population variance S_x^2 of the auxiliary variable x is known in advance. Traditional estimators, such as the usual sample variance, often fail to fully utilise auxiliary information, resulting in inefficiency, particularly in small or complex populations. To address this, ratio- and regression-type estimators have been widely studied, incorporating one or more auxiliary variables to improve precision. Isaki [1] was the first to introduce ratio and regression-based estimators for variance estimation. His estimator was subsequently refined by Prasad and Singh [2], who improved both its precision and bias properties. Later, Arcos et al. [3] presented an advanced ratio-type estimator, which proved to be more accurate and less biased than Isaki [1] method and other traditional estimators. Several studies have focused on the estimation of population variance, including the contributions of Das and Tripath[4], Srivastava and Jhajj [5], Singh et al. [6], Prasad and Singh[7,8], Biradar and Singh[9,10],

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Upadhyaya and Singh[11,12], and Kadilar and Cingi[13,14]. Further work was carried out by Singh and Solanki [15], Singh et al. [16], Mishra and Singh [17,18], Singh and Yadav [19], Audu et al. [20], Singh and Rai [21], and Sharma and Singh [22,23], all of whom contributed significantly to this area of research. In recent years, exponential-type estimators have gained attention due to their flexibility and superior efficiency under certain correlation structures between the study variable and auxiliary variables, as Singh et al. [24] and Grover and Kaur[25].

Motivated by the work of Lu [26], this paper proposes a **novel exponential-type estimator using two auxiliary variables** for estimating unknown population variance. The proposed estimator is designed to reduce bias and MSE compared to existing estimators. This study contributes to the field of survey sampling by providing a robust and efficient method for variance estimation, particularly in situations where auxiliary information is available and can be effectively utilised.

This paper is structured as follows: Section 1 presents the introduction and literature review. Section 2 discusses several existing estimators along with their MSE values. Section 3 introduces the proposed exponential-type estimators based on two auxiliary variables, with bias and MSE derived up to the first-order approximation. Section 4 compares the proposed estimators with the existing ones. Section 5 provides numerical analysis using three real datasets, along with a simulation study to validate the results. Section 6 presents the results and discussion, and finally, Section 7 concludes the paper.

1.1 Notation and Terminology

Let Y and X be the study and auxiliary variables, respectively, of the population of interest, each measured for all population units P_i $(P_1, P_2, P_3 \ldots\ldots\ldots P_N)$. A sample of size n is drawn from this population through simple random sampling without replacement. Let y_i and x_i be the study and auxiliary variables corresponding to the i^{th}sample unit is selected. The population means of these variables are $\bar{Y}$ and $\bar{X}$, respectively. The variances of the study and auxiliary variables are represented by S_y^2 and S_x^2 for the population, and s_y^2 and s_x^2 for the sample.

$\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$, $\bar{X}_1 = \frac{1}{N}\sum_{i=1}^{N} X_{i1}$ and $\bar{X}_2 = \frac{1}{N}\sum_{i=1}^{N} X_{i2}$ are the population means defined for both the study variable and the auxiliary variable.

$S_y^2 = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{Y})^2$,

$S_{x_1}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_{i1} - \bar{X}_1)^2$ and

$S_{x_2}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_{i2} - \bar{X}_2)^2$ are the population variances for the study and auxiliary variables.

$s_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{Y})^2$, $s_{x_1}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_{i1} - \bar{x}_1)^2$ and $s_{x_2}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_{i2} - \bar{x}_2)^2$, are the sample variances.

These notations are used to derive the bias and MSE of the estimators.

Let, $\partial_{400} = \left(\frac{\mu_{400}}{\mu_{200}^2}\right)$ be the population coefficient of kurtosis of y and

$\partial_{210} = \left(\frac{\mu_{210}}{\mu_{200} * \mu_{020}^{\frac{1}{2}}}\right)$ covariance between S_y^2 and $S_{x_1}{}^2$, and

$e_0 = \frac{s_y^2 - S_y^2}{S_y^2}$, $\quad e_1 = \frac{s_{x_1}^2 - S_{x_1}^2}{S_{x_1}^2}$, $\quad e_2 = \frac{s_{x_2}^2 - S_{x_2}^2}{S_{x_2}^2}$,

$E(e_0) = E(e_1) = E(e_2) = 0$,

$E(e_0^2) = \gamma(\partial_{400} - 1), E(e_1^2) = \gamma(\partial_{040} - 1)$,

$E(e_2^2) = \gamma(\partial_{004} - 1)$,

$E(e_0 e_1) = \gamma(\partial_{220} - 1)$,

$E(e_0 e_2) = \gamma(\partial_{202} - 1)$,

$E(e_1 e_2) = \gamma(\partial_{022} - 1)$,

$\gamma = \frac{N-n}{Nn}$, $\partial_{pqr} = \frac{\mu_{pqr}}{\mu_{200}^{\frac{p}{2}}\mu_{020}^{\frac{q}{2}}\mu_{002}^{\frac{r}{2}}}$, and

$\mu_{pqr} = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^p (X_i - \bar{X}_1)^q (X_i - \bar{X}_2)^r$.

In this paper, we have considered

$\partial_{pqr}^* = (\partial_{pqr} - 1)$

2. Existing Estimators for Variance Estimation

The standard estimator for estimating population variance is

$$t_0 = s_y^2 \qquad (2.1)$$

The expressions for the bias and variance of the usual estimator of population variance are as follows:

$$Bias(t_0) = 0 \quad (2.2)$$

$$V(t_0) = S_y^4\, \gamma\, \partial_{400}^* \quad (2.3)$$

Isaki [1], defined a ratio-type estimator of population variance as follows:

$$t_{1r} = s_y^2 \left(\frac{S_x^2}{s_x^2}\right) \quad (2.4)$$

The expressions for the bias and mean squared error of the estimator are, respectively, as follows:

$$Bias(t_{1r}) = S_y^2\gamma[\partial_{040}^* - \partial_{220}^*] \quad (2.5)$$

$$MSE(t_{1r}) = S_y^4\gamma[\partial_{400}^* + \partial_{040}^* - 2\partial_{220}^*] \quad (2.6)$$

The product-type estimator for population variance is defined as:

$$t_{1p} = s_y^2 \left(\frac{s_x^2}{S_x^2}\right) \quad (2.7)$$

The estimator t_{1p} has the following bias and mean squared error:

$$Bias(t_{1p}) = S_y^2\gamma\partial_{220}^* \quad (2.8)$$

$$MSE(t_{1p}) = S_y^4\gamma[\partial_{400}^* + \partial_{040}^* + \partial_{220}^* - 4] \quad (2.9)$$

The exponential ratio-type estimator used for estimating population variance is defined by Singh et al. [24] as:

$$t_{expr} = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right], \quad (2.10)$$

The MSE expression for the estimator t_{expr} is:

$$MSE(t_{expr}) = S_y^4\gamma\left[\partial_{400}^* + \frac{1}{4}\partial_{040}^* - \partial_{220}^* + \frac{1}{4}\right], \quad (2.11)$$

The exponential product-type estimator for the unknown population variance is defined by Singh et al. [24] as follows.

$$t_{\exp(p)} = s_y^2 \exp\left[\frac{s_z^2 - S_z^2}{S_z^2 + s_z^2}\right], \quad (2.12)$$

The MSE of $t_{\exp(p)}$ estimator is obtained as:

$$MSE(t_{\exp(p)}) = S_y^4\gamma\left[\begin{array}{c}\partial_{400}^* + \\ \frac{1}{4}\partial_{004}^* - \partial_{202}^* + \frac{9}{4}\end{array}\right], \quad (2.13)$$

When two auxiliary variables are available, Malik and Singh [27] suggested an estimator for the population as follows:

$$t_{2v} = s_y^2 \left(\frac{S_x^2}{s_x^2}\right)\left(\frac{S_z^2}{s_z^2}\right), \quad (2.14)$$

The MSE of $t_{2v}\, The$ proposed estimator is obtained as:

$$MSE(t_{2v}) = S_y^2\gamma[\partial_{400}^* + \partial_{040}^* + \partial_{004}^* - 2(\partial_{220}^* + \partial_{202}^* - \partial_{022}^*)], \quad (2.15)$$

Singh et al. [24] introduced a general class of exponential estimators, defined as follows.

$$t_{singh} = s_y^2 \left[\begin{array}{c}\alpha \exp\left\{\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right\} + \\ (1-\alpha)\exp\left\{\frac{s_z^2 - S_z^2}{S_z^2 + s_z^2}\right\}\end{array}\right], \quad (2.16)$$

The MSE corresponding to the estimator t_{singh} can be written as:

$$MSE(t_{singh}) = S_y^2\gamma\left[\begin{array}{c}\partial_{400}^* + \frac{\alpha_0^2}{4}\partial_{040}^* + \frac{(1-\alpha_0)^2}{4}\partial_{004}^* - \alpha_0\partial_{220}^* + \\ (1-\alpha_0)\partial_{202}^* - \frac{\alpha_0(1-\alpha_0)}{2}\partial_{022}^*\end{array}\right] \quad (2.17)$$

Using two auxiliary variables, the regression estimator for the population variance of the study variable is formulated as:

$$t_{reg} = s_y^2 + \beta_1(S_x^2 - s_x^2) + \beta_2(S_z^2 - s_z^2), \quad (2.18)$$

The expression for the mean-squared error of t_{reg} is given below

$$MSE(t_{reg}) = S_y^2\gamma\left[\begin{array}{c}\partial_{400}^* + \left(\frac{\beta_1^2}{R_1^2}\right)\partial_{040}^* + \left(\frac{\beta_2^2}{R_2^2}\right)\partial_{004}^* - \\ 2\left(\frac{\beta_1}{R_1}\partial_{220}^* + \frac{\beta_2}{R_2}\partial_{202}^* - \left(\frac{\beta_1\beta_2}{R_1R_2}\right)\partial_{022}^*\right)\end{array}\right], \quad (2.19)$$

Here, β_1 and β_2 represent the estimators of the regression coefficients, defined as:

$$\beta_1 = \frac{S_y^2\partial_{220}^*}{S_x^2\partial_{040}^*}, \beta_2 = \frac{S_y^2\partial_{202}^*}{S_z^2\partial_{004}^*}, R_1 = \frac{S_y^2}{S_x^2}, R_2 = \frac{S_y^2}{S_z^2},$$

3. Proposed Exponential-type Estimators in Two Auxiliary Variables

Auxiliary information makes estimators more efficient and accurate. Following Lu [26], an exponential-type estimator is proposed for estimating the population variance S_y^2 of Y using two auxiliary variables, X_1 and X_2 as follows:

$$t_{rs} = \frac{1}{4}\left[k_1 s_y^2 + k_2(S_{x_1}^2 - s_{x_1}^2) + k_3(S_{x_2}^2 - s_{x_2}^2)\right]\left[\exp\left\{\frac{S_{x_1}^2 - s_{x_1}^2}{S_{x_1}^2 + s_{x_1}^2}\right\} + \exp\left\{\frac{s_{x_1}^2 - S_{x_1}^2}{S_{x_1}^2 + s_{x_1}^2}\right\}\right]\left[\exp\left\{\frac{S_{x_2}^2 - s_{x_2}^2}{S_{x_2}^2 + s_{x_2}^2}\right\} + \exp\left\{\frac{s_{x_2}^2 - S_{x_2}^2}{S_{x_2}^2 + s_{x_2}^2}\right\}\right], \quad (3.1)$$

Expressing equation (3.1) in terms of $e_i's\ (i = 0,1,2)$, we have,

$$t_{rs} = \frac{1}{4}\left[k_1 S_y^2(1+e_0) + k_2\left(S_{x_1}^2 - S_{x_1}^2(1+e_1)\right) + k_3\left(S_{x_2}^2 - S_{x_2}^2(1+e_2)\right)\right]\left[\exp\left\{\frac{S_{x_1}^2 - S_{x_1}^2(1+e_1)}{S_{x_1}^2 + S_{x_1}^2(1+e_1)}\right\} +\right.$$

$\exp\left\{\frac{S_{x_1}^2(1+e_1)-S_{x_1}^2}{S_{x_1}^2+S_{x_1}^2(1+e_1)}\right\}\Big]\left[\exp\left\{\frac{S_{x_2}^2-S_{x_2}^2(1+e_2)}{S_{x_2}^2+S_{x_2}^2(1+e_2)}\right\}+\exp\left\{\frac{S_{x_2}^2(1+e_2)-S_{x_2}^2}{S_{x_2}^2+S_{x_2}^2(1+e_2)}\right\}\right],$ (3.2)

It can be written as

$t_{rs}=\left[k_1S_y^2(1+e_0)-k_2S_{x_1}^2e_1-k_3S_{x_2}^2e_2\right]\left\{1+\frac{1}{8}e_1^2+\frac{1}{8}e_2^2\right\},$ (3.3)

Expanding equation (3.3), we obtain the expression:

$t_{rs}=k_1S_y^2+k_1e_0S_y^2+\frac{1}{8}e_1^2k_1S_y^2+\frac{1}{8}e_2^2k_1S_y^2-k_2e_1S_{x_1}^2-k_3e_2S_{x_2}^2,$ (3.4)

By subtracting S_y^2 from both sides and taking expectations, the bias of the proposed estimator is obtained as:

$Bias(t_{rs})=(k-1)S_y^2+k_1S_y^2\left[\frac{1}{8}E(e_1^2)+\frac{1}{8}E(e_2^2)\right],$ (3.5)

$Bias(t_{rs})=(k_1-1)S_y^2+k_1S_y^2\left[\frac{1}{8}(\partial_{040}^*)+\frac{1}{8}(\partial_{004}^*)\right],$ (3.6)

We can write equation (3.4) as follows after subtracting S_y^2,

$\left(t_{rs}-S_y^2\right)\cong(k_1-1)S_y^2+k_1e_0S_y^2+\frac{1}{8}e_1^2k_1S_y^2+\frac{1}{8}e_2^2k_1S_y^2-k_2e_1S_{x_1}^2-k_3e_2S_{x_2}^2,$ (3.7)

By squaring both sides and taking expectations, we obtain the following expression:

$\left(t_{rs}-S_y^2\right)^2=S_y^2+k_1^2\left(S_y^4+S_y^2e_0^2+\frac{1}{4}S_y^4e_1^2+\frac{1}{4}S_y^4e_2^2\right)+k_2^2\left(S_{x_1}^4e_1^2\right)+k_3^2\left(S_{x_2}^4e_2^2\right)-2k_1\left(S_y^4+\frac{1}{8}S_y^4e_2^2+\frac{1}{8}S_y^4e_1^2\right)-2k_1k_2\left(S_y^2S_{x_1}^2e_0e_1\right)-2k_1k_3\left(S_y^2S_{x_2}^2e_0e_2\right)+2k_2k_3(S_{x_1}^2S_{x_2}^2e_1e_2),$ (3.8)

Equation (3.8) can be written as:

$MSE(t_{rs})=S_y^4+k_1^2A+k_2^2B+k_3^2C-2k_1D-2k_1k_2E-2k_1k_3G+2k_2k_3F,$ (3.9)

Where,

$A=S_y^4+S_y^4\gamma\left[(\partial_{400}^*)+\frac{1}{4}(\partial_{040}^*)+\frac{1}{4}(\partial_{004}^*)\right],$

$B=S_{x_1}^4\gamma\partial_{004}^*,$

$C=S_{x_2}^4\gamma\partial_{004}^*,$

$D=S_y^4+S_y^4\gamma\frac{1}{8}[\partial_{040}^*+\partial_{004}^*],$

$E=S_y^2S_{x_1}^2\gamma\partial_{220}^*,$

$F=S_{x_1}^2S_{x_2}^2\gamma\partial_{022}^*,$

$G=S_y^2S_{x_2}^2\gamma\partial_{202}^*.$

By taking partial derivatives of equation (3.9) with respect to k_1, k_2 and k_3, we get the optimum values of the constant respectively as

$k_{1(opt)}=\frac{D(BC-F^2)}{\delta}$

$k_{2(opt)}=\frac{D(CE-FG)}{\delta},$

$k_{3(opt)}=\frac{D(BG-EF)}{\delta},$

Where,

$\delta=ABC-AF^2-CE^2-BG^2+2EFG.$

Substituting optimum values of k_1, k_2 and k_3 into equation (3.9) we get the minimum MSE as follows

$Min.MSE(t_{rs})=S_y^4+\left(\frac{D^2F^2-BCD^2}{ABC-AF^2-CE^2-BG^2+2EFG}\right),$ (3.10)

4. Theoretical Comparison of the Proposed Estimator with Existing Estimators

(i) The proposed estimator t_{rs} will be more efficient than the estimator t_0 if the following condition holds:

$Min.MSE(t_{rs})-MSE(t_0)<0,$ (4.1)

$S_y^4+\left(\frac{D^2F^2-BCD^2}{ABC-AF^2-CE^2-BG^2+2EFG}\right)-S_y^4\,\gamma\,\partial_{400}^*<0,$ (4.2)

(ii) The proposed estimator t_{rs} will be more efficient than the estimator t_r if the following condition holds:

$Min.MSE(t_{rs})-MSE(t_r)<0,$ (4.3)

$S_y^4+\left(\frac{D^2F^2-BCD^2}{ABC-AF^2-CE^2-BG^2+2EFG}\right)-S_y^4\gamma[\partial_{400}^*+\partial_{040}^*-2\partial_{220}^*]<0,$ (4.4)

(iii) The proposed estimator t_{rs} will be more efficient than the estimator t_p if the following condition holds:

$Min.MSE(t_{rs})-MSE(t_p)<0,$ (4.5)

$S_y^4+\left(\frac{D^2F^2-BCD^2}{ABC-AF^2-CE^2-BG^2+2EFG}\right)-S_y^4\gamma[\partial_{400}^*+\partial_{040}^*+\partial_{220}^*-4]<0,$ (4.6)

(iv) The proposed estimator t_{rs} will be more efficient than the estimator $t_{\exp(r)}$ if the following condition holds:

$Min.MSE(t_{rs})-MSE(t_{\exp(r)})<0,$ (4.7)

$S_y^4+\left(\frac{D^2F^2-BCD^2}{ABC-AF^2-CE^2-BG^2+2EFG}\right)-S_y^4\gamma\left[\partial_{400}^*+\frac{1}{4}\partial_{040}^*-\partial_{220}^*+\frac{1}{4}\right]<0,$ (4.8)

(v) The proposed estimator t_{rs} will be more efficient than the estimator $t_{\exp(p)}$ if the following condition holds:

$Min. MSE(t_{rs}) - MSE(t_{\exp(p)}) < 0,$ (4.9)

$S_y^4 + \left(\frac{D^2F^2 - BCD^2}{ABC - AF^2 - CE^2 - BG^2 + 2EFG}\right) - S_y^4\gamma\left[\partial_{400}^* + \frac{1}{4}\partial_{004}^* - \partial_{202}^* + \frac{9}{4}\right] < 0,$ (4.10)

(vi) The proposed estimator t_{rs} will be more efficient than the estimator $t_{2vratio}$ if the following condition holds:

$Min. MSE(t_{rs}) - MSE(t_{2vratio}) < 0,$ (4.11)

$S_y^4 + \left(\frac{D^2F^2 - BCD^2}{ABC - AF^2 - CE^2 - BG^2 + 2EFG}\right) - S_y^2\gamma[\partial_{400}^* + \partial_{040}^* + \partial_{004}^* - 2(\partial_{220}^* + \partial_{202}^* - \partial_{022}^*)] < 0,$ (4.12)

(vii) The proposed estimator t_{rs} will be more efficient than the estimator t_{singh} if the following condition holds:

$Min. MSE(t_{rs}) - MSE(t_{singh}) < 0,$ (4.13)

$S_y^4 + \left(\frac{D^2F^2 - BCD^2}{ABC - AF^2 - CE^2 - BG^2 + 2EFG}\right) - S_y^2\gamma\left[\partial_{400}^* + \frac{\alpha_0^2}{4}\partial_{040}^* + \frac{(1-\alpha_0)^2}{4}\partial_{004}^* - \alpha_0\partial_{220}^* + (1-\alpha_0)\partial_{202}^* - \frac{\alpha_0(1-\alpha_0)}{2}\partial_{022}^*\right] < 0,$ (4.14)

(viii) The proposed estimator t_{rs} will be more efficient than the estimator t_{reg} if the following condition holds:

$Min. MSE(t_{rs}) - MSE(t_{reg}) < 0,$ (4.15)

$S_y^4 + \left(\frac{D^2F^2 - BCD^2}{ABC - AF^2 - CE^2 - BG^2 + 2EFG}\right) - S_y^2\gamma\left[\partial_{400}^* + \left(\frac{\beta_1^2}{R_1^2}\right)\partial_{040}^* + \left(\frac{\beta_2^2}{R_2^2}\right)\partial_{004}^* - 2\left(\frac{\beta_1}{R_1}\partial_{220}^* + \frac{\beta_2}{R_2}\partial_{202}^* - \left(\frac{\beta_1\beta_2}{R_1R_2}\right)\partial_{022}^*\right)\right] < 0,$ (4.16)

5. Numerical Analysis

We have used three real-world datasets, each chosen by simple random sampling. The parameters of these datasets are presented in Table 1. We examine the performance of various estimators by comparing them through a metric known as percentage relative efficiency (PRE).

The equation for relative efficiency is expressed as

$$PRE(t_i) = \frac{MSE(t_0)}{MSE(t_i)} \times 100$$

Data set 1, taken from Murthy [28], consists of the workshop output as the study variable (Y), with fixed capital (X_1) and the number of workers (X_2) as auxiliary variables.

Data set 2 is from Cochran [29], which includes food cost as the study variable (Y), while family size (X_1) and family income (X_2) serves as an auxiliary variable. Finally, data set 3, reported by Singh [30], considers the estimated fish catch in 1995 as the main variable (Y), with the estimated fish catch in 1994 (X_1) and in 1993 (X_2) as the supporting variables.

Table 1: Descriptive table for the data sets

Parameter	Population 1	Population 2	Population 3
N	80	33	69
n	24	8	25
γ	0.02916667	0.09469697	0.02550725
$\bar{Y}$	5182.637	27.49091	4514.899
$\bar{X}_1$	285.125	3.727273	4954.453
$\bar{X}_2$	1126.463	72.54545	4591.072
S_y^2	3369642	102.6327	37199578
$S_{x_1}^2$	73132.09	2.329545	49829270
$S_{x_2}^2$	715055.8	111.8807	39881874
C_y	0.3541939	0.3685139	1.350893
C_{x_1}	0.9484593	0.4094911	1.424781
C_{x_2}	0.7506772	0.1458033	1.375541
ρ_{yx_1}	0.9149811	0.432738	0.9601401

ρ_{yx_2}	0.9413055	0.2521603	0.956907
$\rho_{x_1x_2}$	0.9884207	-0.06598959	0.972902
∂_{400}	1.210342	4.382758	6.544991
∂_{040}	2.491817	1.245254	8.69773
∂_{004}	1.79522	1.015036	8.698631
∂_{220}	1.294326	0.388943	7.053057
∂_{202}	1.193045	1.218625	7.190267
∂_{022}	2.093142	0.4465917	8.523088

Table 2: PRE of the existing and proposed estimators

Estimators	Population 1	Population 2	Population 3
t_0	100	100	100
t_r	108.6964	90.3638	575.8359
t_p	22.4484	81.2879	121.3912
$t_{\exp(r)}$	153.4078	96.2159	341.5313
$t_{\exp(p)}$	42.4355	153.6628	229.4723
$t_{2v\,ratio}$	25.7032	101.4270	52.3562
t_{singh}	136.9788	106.2171	873.1611
t_{reg}	101.6870	139.8830	103.7552
t_{rs}	553.9061	201.2975	1533.3547

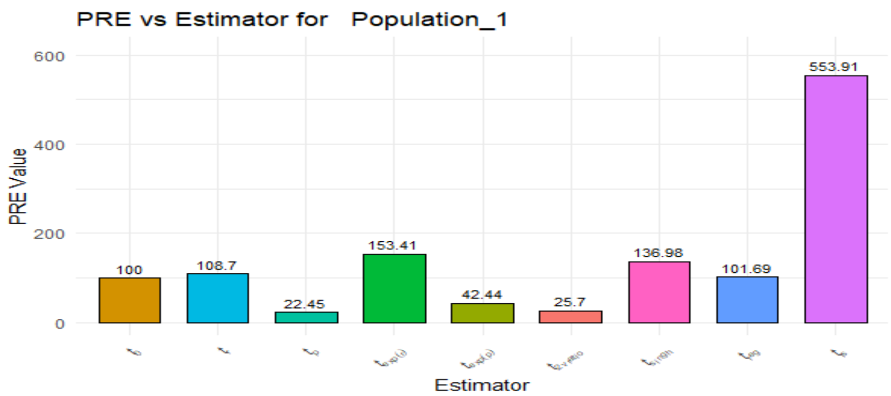

Figure 1: Bar diagram of PRE value and estimator for population 1.

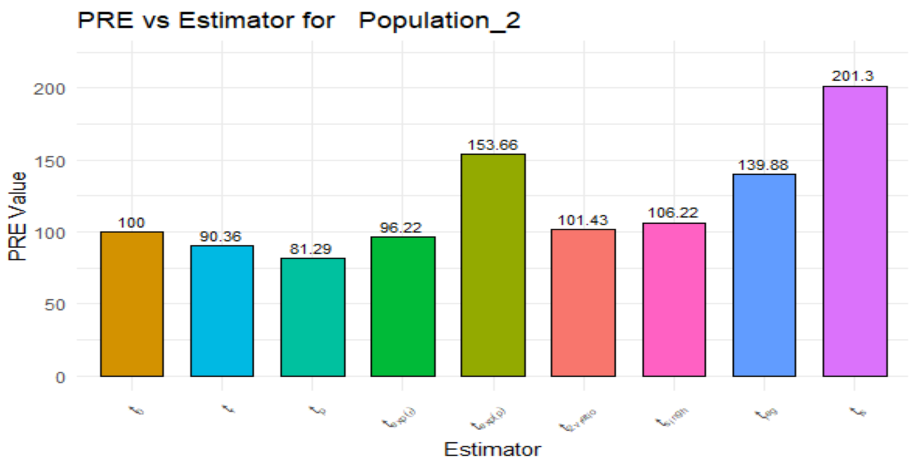

Figure 2: Bar diagram of PRE values and estimator for population 2.

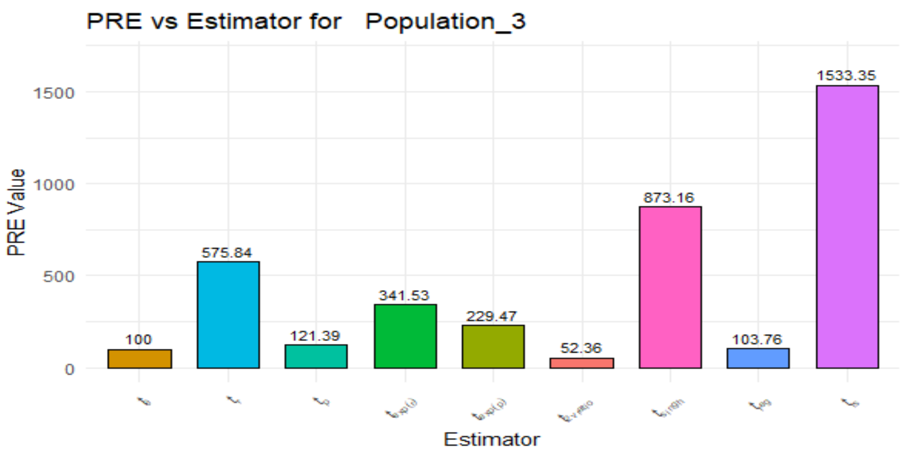

Figure 3: Bar diagram of PRE values and estimator for population 3.

5.1 Simulation Study

This section presents a simulation study to evaluate the MSE and PRE of the proposed estimators. The simulation is carried out using the following steps:

1. A bivariate normal population of size N=1000 is generated with the following parameters:
   μ_x = 9, σ_x = 4;
   μ_y = 25, σ_y = 9;
   $\mu_z = 50$, $\sigma_z = 25$,
   and correlation coefficients ρ = 0.75, 0.80, and 0.90.
2. A sample of sizes n = 100, 200 and 400 is selected from the simulated population.
3. For each sample, the sample mean, sample variance, and values of both the proposed and existing estimators of the population mean are calculated.
4. The entire simulation was repeated 10,000 times to calculate MSEs; the average of these 10,000 values represents the MSE of each estimator for the population mean.

$$MSE(t_0) = \frac{1}{10{,}000}\sum_{i=1}^{N}(t_i - E(t_i))^2$$

5. And the percent relative efficiency of each estimator $t_h$relative to a reference estimator t_0 is also obtained.

Table 3. MSE and PRE-Table of the proposed estimators and existing estimators in the case of the simulation study.

Estimators	N=1000, n=200		N=1000, n=350		N=1000, n=400	
	MSE	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
t_0	0.23784	100	0.13704	100	0.11084	100
t_r	0.23387	101.6975	0.13033	105.14847	0.10109	109.6449
t_p	0.64102	37.10337	0.39554	34.64631	0.30257	36.63285
$t_{\exp(r)}$	0.2015	118.0347	0.1101	124.46866	0.08867	125.0028
$t_{\exp(p)}$	0.55526	42.83399	0.3403	40.27035	0.26712	41.49446
$t_{2v\,ratio}$	0.43761	54.34976	0.2577	53.17811	0.2101	52.75583
t_{singh}	0.1951	121.9067	0.10425	131.45324	0.08638	128.3167
t_{reg}	0.17829	133.4006	0.09555	143.42229	0.07975	138.9843
t_{rs}	0.16493	144.2066	0.08852	154.81247	0.07401	149.7636

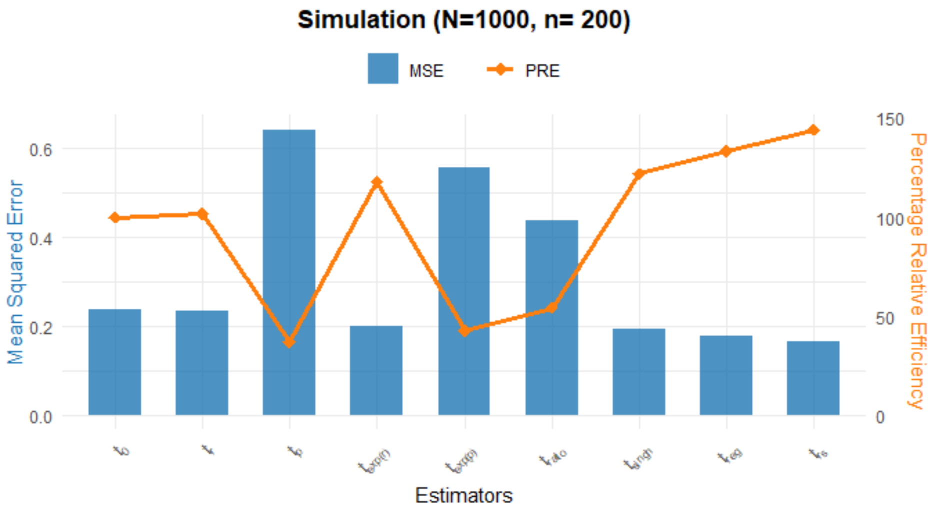

Figure 4: Bar and Line diagram of MSE and PRE values for simulation at n=200.

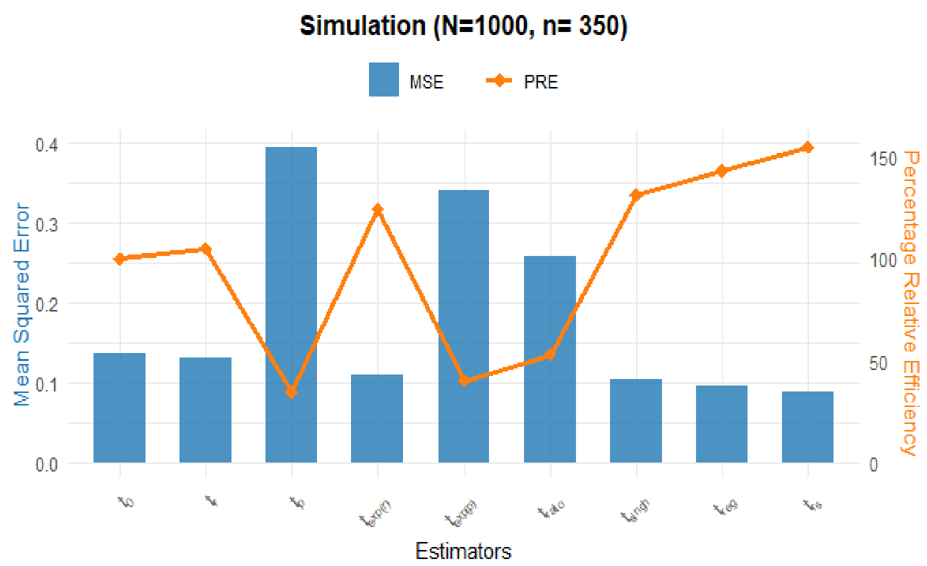

Figure 5: Bar and Line diagram of MSE and PRE values for simulation at n=350.

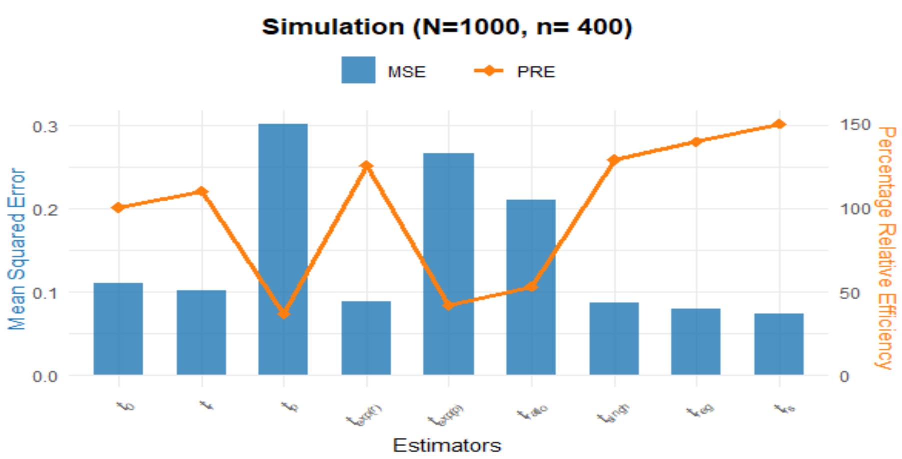

Figure 6: Bar and Line diagram of MSE and PRE values for simulation at n=400.

6. Results and discussion

In this paper, we have developed an exponential-type estimator that employs two auxiliary variables to estimate the unknown population variance. In order to evaluate the performance of the proposed estimator in comparison to other existing estimators, we have derived the Bias and MSE. The estimators' theoretical comparison has been discussed in order to achieve a more precise result.

In the numerical analysis, three real datasets are considered to evaluate the performance of the proposed estimator. The results, measured through PRE, clearly indicate that the proposed estimator consistently outperforms the existing estimators. For example, in Population 1, the proposed estimator t_{rs}achieved a PRE of 553.90 compared to the baseline estimator, as measured by the sample variance. Similarly, in Population 2, it attained a PRE of 201.29, while in Population 3, it demonstrated with PRE of 1533.35. These findings highlight the remarkable efficiency gain achieved by incorporating two auxiliary variables through an exponential-type structure. The bar diagrams for each population visually reinforce these findings, clearly showing t_{rs} as the tallest bar in each case, thus confirming its better performance.

To further validate these results, a simulation study has been conducted using a hypothetical population of size 1000 with different sample sizes of 200, 350, and 400. The simulation was repeated 10,000 times to ensure reliable estimates of MSE and PRE. The findings revealed that the proposed estimator consistently produced the lowest MSE and highest PRE across all sample sizes. For instance, at n=200, the estimator achieved an MSE of 0.1649 with a PRE of 144.20, outperforming even the regression estimator in two auxiliary variables. At larger sample sizes such as n=350 and n=400, the superiority of the proposed estimator persisted, with PRE values of 154.81 and 149.76, respectively. The combined bar and line diagrams for the simulation scenarios visually illustrate this superiority, with the shortest bar (lowest MSE) and the highest point (highest PRE) always corresponding to the proposed estimator t_{rs}.

7. Conclusion

In conclusion, the study establishes that the proposed exponential-type estimatort_{rs} is a highly efficient and reliable alternative to traditional estimators. Both real datasets and simulation experiments consistently confirm its superiority, showing substantial reductions in MSE and impressive gains in PRE across different populations, correlation levels, and sample sizes. The numerical analysis, supported by graphical illustrations, provides strong empirical evidence that the estimator is not only theoretically sound but also practically robust. By effectively utilising auxiliary information, the proposed estimator enhances the precision of population variance estimation, making it a powerful and versatile tool in the field of survey sampling.

References

[1] Isaki, C. T. (1983). Variance estimation using auxiliary information. *Journal of the American Statistical Association*, *78*(381), 117–123.

[2] Prasad, B., & Singh, H. P. (1990). Some improved ratio type estimators of finite population variance in sample surveys. *Communications in Statistics—Theory and Methods*, *19*(4), 1127–1139.

[3] Arcos, A., Barrios, E., & Sarabia, J. M. (2005). Incorporating the auxiliary information available in variance estimation. *Applied Mathematics and Computation*, *160*(2), 387–399.

[4] Das, A. K., & Tripathi, T. P. (1978). Use of auxiliary information in estimating the finite population variance. *Sankhyā C, 40*(2), 139–148.

[5] Srivastava, S. K., & Jhajj, H. S. (1980). A class of estimators using auxiliary information for estimating finite population variance. *Sankhyā C, 42*(1–2), 87–96.

[6] Singh, H. P., Upadhyaya, L. N., & Namjoshi, U. D. (1988). Estimation of finite population variance. *Current Science*, *57*(24), 1331–1334.

[7] Prasad, B., & Singh, H. P. (1990). Some improved ratio-type estimators of finite population variance in sample surveys. *Communications in Statistics—Theory and Methods*, *19*(3), 1127–1139.

[8] Prasad, B., & Singh, H. P. (1992). Unbiased estimators of finite population variance using auxiliary information in sample surveys. *Communications in Statistics—Theory and Methods*, *21*(5), 1367–1376.

[9] Biradar, R. S., & Singh, H. P. (1994). An alternative to ratio estimator of population variance. *Assam Statistical Review, 8*(2), 18–33.
[10] Biradar, R. S., & Singh, H. P. (1998). Predictive estimation of finite population variance. *Calcutta Statistical Association Bulletin, 48*(191–192), 229–235.
[11] Upadhyaya, L. N., & Singh, H. P. (1999). An estimator of population variance that utilizes the kurtosis of an auxiliary variable in sample surveys. *Vikram Mathematical Journal, 19*, 14–17.
[12] Upadhyaya, L. N., & Singh, H. P. (2001). Estimation of the population standard deviation using auxiliary information. *American Journal of Mathematical and Management Sciences, 21*(3–4), 345–358.
[13] Kadilar, C., & Cingi, H. (2006). Ratio estimators for the population variance in simple and stratified random sampling. *Applied Mathematics and Computation, 173*(2), 1047–1059.
[14] Kadilar, C., & Cingi, H. (2007). Improvement in variance estimation in simple random sampling. *Communications in Statistics—Theory and Methods, 36*, 2075–2081.
[15] Singh, H. P., & Solanki, R. S. (2013). A new procedure for variance estimation in simple random sampling using auxiliary information. *Statistical Papers, 54*(2), 479–497.
[16] Singh, H. P., Pal, S. K., & Solanki, R. S. (2013). Improved estimation of finite population variance using quartiles. *Istatistik: Journal of the Turkish Statistical Association, 6*(3), 166–121.
[17] Mishra, P., & Singh, R. (2016). Variance estimation using arithmetic mean, geometric mean, and harmonic mean under simple random sampling. *Journal of Scientific Research, 60*(1–2), 97–109.
[18] Mishra, P., & Singh, R. (2016). Variance estimation using quality characteristic. *Merit Research Journal of Interdisciplinary Studies, 5*(2), 143–149.
[19] Singh, R., & Yadav, S. K. (2024). Efficient estimators of population coefficient of variation under simple random sampling using single auxiliary variable. *Journal of Statistical and Applied Probability Letters, 11*(3), 109–121.
[20] Audu, A., Singh, R., Khare, S., & Duran, N. S. (2022). Almost unbiased variance estimators under the simultaneous influence of non-response and measurement errors. *Journal of Statistical Theory and Practice.*
[21] Singh, R., & Rai, S. (2021). Estimating population variance using the median as an auxiliary information. *Journal of Scientific Research, 65*(5), 222–225.
[22] Sharma, P., & Singh, R. (2013). A generalised class of estimators for finite population variance in the presence of measurement errors. *Journal of Modern Applied Statistical Methods, 12*(2), 231–241.
[23] Sharma, P., & Singh, R. (2014). Improved dual to variance ratio type estimator for population variance. *Chilean Journal of Statistics, 5*(2), 45–54.
[24] Singh, R., Chauhan, P., Sawan, N., & Smarandache, F. (2009). Improved exponential estimator for population variance using two auxiliary variables. *Studies in Statistical Inference, Sample Technique and Demography, 36*.
[25] Grover, L. K., & Kaur, A. (2011). Ratio-type exponential estimators of population mean under linear transformation of auxiliary variable: Theory and methods. *South African Statistical Journal, 45*(2), 205–230.
[26] Lu, J. (2017). Efficient estimator of a finite population mean using two auxiliary variables and numerical application in agricultural, biomedical, and power engineering. *Mathematical Problems in Engineering, 2017*(1), Article 8704734.
[27] Malik, S., & Singh, R. (2014). A family of estimators for population variance using two auxiliary variables. *Elixir Statistics, 66*, 20278–20286.
[28] Murthy, M. N. (1967). *Sampling: Theory and Methods*. Statistical Publishing Society.
[29] Cochran, W. G. (1977). *Sampling Techniques*. Wiley.
[30] Singh, S. (2003). *Advanced Sampling Theory with Applications: How Michael "Selected" Amy* (Vol. 2). Springer.