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Estimation and Some Statistical Properties of the hybrid Weibull Inverse Burr Type X Distribution with Application to Cancer Patient Data

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ABSTRACT

In this paper, the Burr type X distribution is represented as a mixture with the hybrid Weibull family, called hybrid hybrid Hybrid Weibull Inverse Burr Type X (HWIBX) distribution in order to justify its usefulness in modeling statistical regularities in extreme values recorded in non-stationary streams of media events, the basic new distribution functions are presented in addition to several statistical and mathematical properties of the hybrid distribution as well as estimating the model parameters by three methods, namely maximum likelihood, least squares, and weighted least squares with Conduct Monte Carlo simulations for the three methods. Finally, these results are illustrated through an example of matching the hybrid distribution with data representing the values Bladder cancer and survival times (given in years) of a group comprising 46 patients treated with chemotherapy alone and knowing the results of the improvement of this expansion through comparison with some other distributions using some Statistical metrics.

1. Introduction

The need for an accurate graphical representation of temporal events and continuous changes in data values that showed the basic distributions such as the exponential distribution, the normal distribution, or other distributions is futile in the way that these changes, mutations, and the appearance of extreme values appear. Therefore, the need to develop the basic distributions has arisen, and for this purpose many have demonstrated Researchers in the field of mathematical statistics have taken the lead in creating new distributions by creating families of continuous and discrete statistical distributions to give more flexibility and accurate reading of the data. One of the important methods that have been used to generate families of continuous distributions is the T-X method, which was proposed by Al- Alzaatara et al in 2013 to generate families of new statistical distributions [1] by form:

$$F(x,\psi)=\int_{0}^{\phi(G(x,\psi))} r(x)\,dx \qquad (1)$$

where r(x) is any pdf, G(x,ψ) is the baseline distribution, and ϕ(G(x,ψ)) is a CDF, and ψ is the parameters of distribution.

Examples of these families include: MOW-G family [2], WW-G by [3], Right truncated Xgamma-G by [4], OL-G by [5], and HOE-Φ by [6], To find the hybrid distribution, we will rely on the Weibull distribution using the T-X

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method and a hybrid $\phi(G(x,\psi))$ function that combines $G(x,\psi)$ [7], [8] and $-log[1- G(x,\psi)]$ [9] in the form:

$$\phi(G(x,\psi)) = -G(x,\psi)log[1- G(x,\psi)] \quad (2)$$

So that the $\phi(G(x,\psi))$ sufficient the conditions:

a. $\phi(G(x,\psi)) = -G(x,\psi)log[1- G(x,\psi)] \in [a,b], -\infty < a < b < \infty$

b. $\phi(G(x,\psi)) = -G(x,\psi)log[1- G(x,\psi)]$ is non-decreasing, differentiable, and monotonically

c. $\phi(G(x,\psi)) = -G(x,\psi)log[1- G(x,\psi)] \to 0, as\ x \to 0$, and

d. $\phi(G(x,\psi)) = -G(x,\psi)log[1- G(x,\psi)] \to 1, as\ x \to \infty$

Using the PDF Weibull distribution of the form $r(x) = \delta\theta x^{\theta-1}e^{-\delta x^{\theta}}$ in Equation 1, in addition to substituting Equation 2 into Equation 1, we obtain CDF of the hybrid family function (HWG) of the form:

$$F(x,\delta,\theta,\psi) = \int_0^{-G(x,\psi)log[1- G(x,\psi)]} \delta\theta x^{\theta-1}e^{-\delta x^{\theta}}\,dx \quad (1)$$

$$F(x,\psi) = 1 - e^{-\delta[-G(x,\psi)log[1- G(x,\psi)]]^{\theta}} \quad (2)$$

By deriving equation 4, we obtain the family pdf function of the form:

$$f(x,\delta,\theta,\psi) = \delta\theta\, g(x,\psi)\left[\frac{G(x,\psi)}{1-G(x,\psi)} - \log(1-G(x,\psi))\right] \times [-G(x,\psi)log[1 - G(x,\psi)]]^{\theta-1}e^{-\delta[-G(x,\psi)log[1- G(x,\psi)]]^{\theta}} \quad (3)$$

where $\delta,\theta > 0$ are shape parameters for the family, and $x > 0$ is any random variable.

The study aims to generate a continuous statistical distribution called the hybrid Weibull inverse Burr-X (HWIBX) distribution with four parameters representing two parameters for the hybrid family and two parameters for the basic distribution, in addition to finding some basic characteristics of the distribution and estimating its parameters in three ways, as well as performing a simulation of the estimated parameters of the distribution for the three methods to determine the bias in the parameter values, and finally conducting a practical application on two types of cancer data to demonstrate the efficiency of the hybrid distribution.

The study consisted of five parts, the first part represented finding the basic functions of the new distribution, while the second part included finding some mathematical properties of the distribution as well as expanding the pdf and CDF functions of the hybrid distribution. The third part included estimating the model parameters using MLE, LSE, and WLSE methods. While the goal of the third part was to conduct a Monte Carlo simulation of the parameters estimated in the third part, and finally the fifth part included conducting a practical application on two types of cancer data in which programming in the R language was used.

2. Hybrid Weibull Inverse Burr-X (HWIBX) distribution

Let x represent the random variable, the CDF functions of Inverse Burr X with two-parameter is [10]:

$$G(x,\alpha,\gamma) = 1 - \left(1 - e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma} \quad (4)$$

And pdf function has form [10]:

$$g(x,\alpha,\gamma) = 2\alpha^2\gamma x^{-3}e^{-\frac{\alpha^2}{x^2}}\left(1 - e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma-1} \quad (5)$$

By substituting Equation 6 into Equation 4, we obtain the CDF function for the new distribution HWIBX in the form:

$$F(x,\delta,\theta,\alpha,\gamma) = 1 - e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}}, x \geq 0 \quad (6)$$

where $\delta, \theta, \alpha, \gamma > 0$ are shape parameters of HWIBX.

Using programming in the R language, the CDF function is drawn for the HWIBX distribution for different parameter values, as shown in Figure 1.

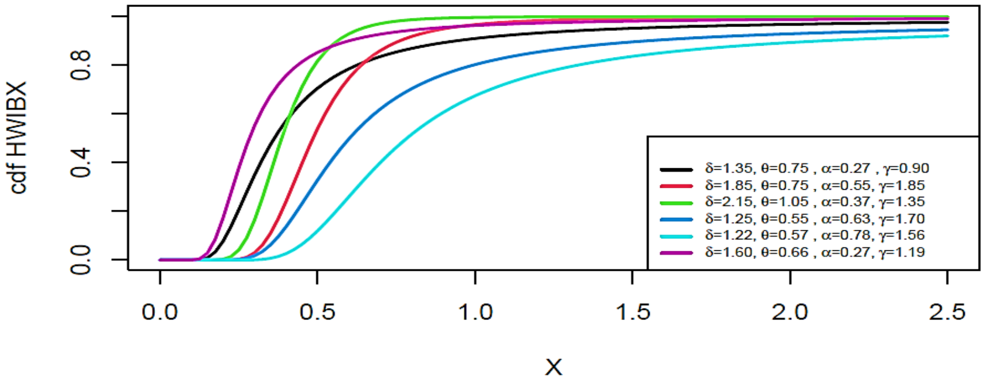

Figure 1. CDF function of HWIBX distribution for different parameter values

To find the pdf function for the HWIBX distribution, equation 8 is derived or equation 6 and 7 are substituted into equation 5 in order to obtain the equation in the form:

$$f(x,\delta,\theta,\alpha,\gamma) = \delta\theta\, 2\alpha^2\gamma x^{-3} e^{-\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma-1}\left[\frac{1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}}{\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}} - \log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right] \times$$
$$\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right) log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta-1} e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}} \quad (7)$$

Using programming in the R language, the pdf function is drawn for the HWIBX distribution for different parameter values, as shown in Figure 2.

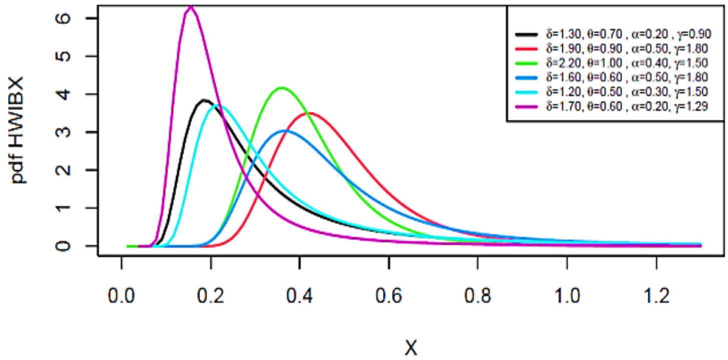

Figure 2. pdf function of HWIBX distribution for different parameter values

The Survival function has the relationship [11,12]:

$$S(x)_{HWIBX} = 1 - F(x,\delta,\theta,\alpha,\gamma) \quad (8)$$

Thus, by substituting equation (8) into the aforementioned equation, we obtain:

$$S(x)_{HWIBX} = e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}} \quad (9)$$

Using programming in the R language, the Survival function is drawn for the HWIBX distribution for different parameter values, as shown in Figure 3.

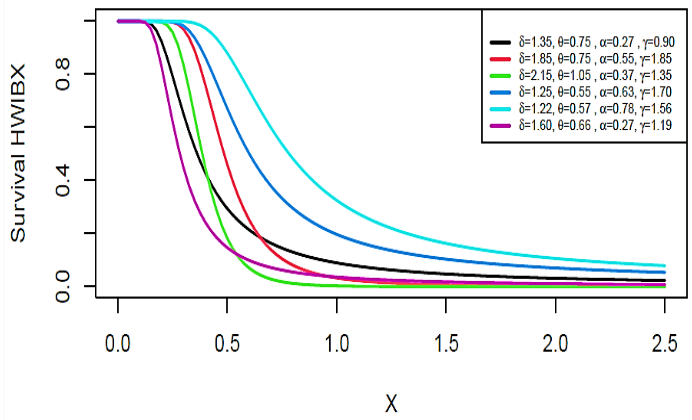

Figure 3. Survival function of HWIBX distribution for different parameter values

The Hazard function, which holds significant importance, particularly in relation to matters of life, has garnered considerable interest from researchers. They have concentrated their efforts on identifying statistical distributions of various kinds for this function. Consequently, the following relationship can be utilized to derive it [13,14]:

$$h(x)_{HWIBX} = \frac{f(x,\delta,\theta,\alpha,\gamma)}{S(x)_{HWIBX}} \tag{10}$$

Thus, by substituting equation (9) and (11) into the previous equation, we obtain:

$$h(x)_{HWIBX} = \delta\theta\, 2\alpha^2\gamma x^{-3} e^{-\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma-1}\left[\frac{1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}}{\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}} - \log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right] \times\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right) log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta-1} \tag{11}$$

Using programming in the R language, the Hazard function is drawn for the HWIBX distribution for different parameter values, as shown in Figure 3.

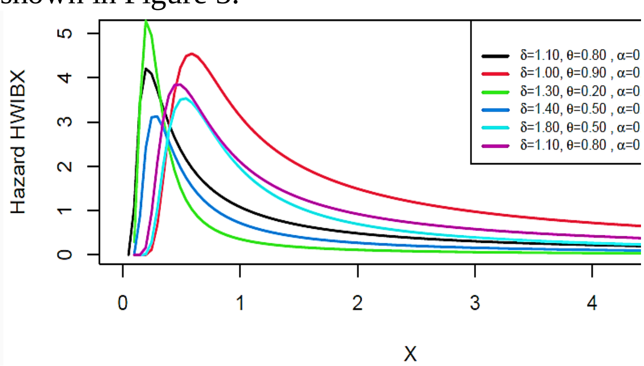

Figure 4. Hazard function of HWIBX distribution for different parameter values

From Figures 2 and 3, it is observed that when drawing the CDF functions and remaining in a row, the values fall within the interval [0,1], which fulfills the condition of the CDF function. However, Figure 4 shows that drawing the functions takes different shapes, resembling an inverted bathtub, which is what It indicates the flexibility of the new distribution.

3. Mathematical Properties of HWIBX distribution

3.1. Useful representations CDF, CDF^d, and pdf

In order to obtain expanded (simplified) CDF and PDF functions, exponential and logarithmic function expansions and binomial series expansions are used. First the CDF function is expanded as follows [5]:

$$e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}} = \sum_{i=0}^{\infty}\frac{(-1)^{i(1+\theta)}}{i!}\,\delta^i\left[\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{i\theta}$$

By using logarithm expansion for $\left(\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)^{i\theta}$ by the form:

$$\left(\log\left[1-1+\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]\right)^{i\theta}=\sum_{j=0}^{\infty}(-1)^{j}D_{i\theta,j}\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)^{j+i\theta}$$

where $D_{i\theta,j}=j^{-1}\sum_{m=1}^{j}\frac{m(i\theta+1)-j}{m+1}D_{i\theta,j-m}$, $j\geq 1, D_{i\theta,0}=1$

$$\therefore F(x,\delta,\theta,\alpha,\gamma)=1-\sum_{i=j=0}^{\infty}\frac{(-1)^{i(1+\theta)+j}}{i!}D_{i\theta,j}\,\delta^{i}\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)^{j+2i\theta}$$

Now using binomial series expansion for $\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)^{j+2i\theta}$ we get:

$$\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)^{j+2i\theta}=\sum_{k=0}^{\infty}(-1)^{k}\binom{j+2i\theta}{k}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{k\gamma}$$

Again using binomial series expansion for $\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{k\gamma}$ we get:

$$\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{k\gamma}=\sum_{p=0}^{\infty}(-1)^{p}\binom{k\gamma}{p}e^{-p-\frac{\alpha^2}{x^2}}$$

Finally the CDF of HWIBX distribution has the form:

$$F(x,\delta,\theta,\alpha,\gamma)=1-\sum_{i=j=k=p=0}^{\infty}\frac{(-1)^{i(1+\theta)+j+k+p}}{i!}D_{i\theta,j}\binom{j+2i\theta}{k}\binom{k\gamma}{p}\,\delta^{i}e^{-p-\frac{\alpha^2}{x^2}}$$

Let

$\Omega=\sum_{i=j=k=p=0}^{\infty}\frac{(-1)^{i(1+\theta)+j+k+p}}{i!}D_{i\theta,j}\binom{j+2i\theta}{k}\binom{k\gamma}{p}\,\delta^{i}$

then:

$$F(x,\delta,\theta,\alpha,\gamma)=1-\Omega e^{-p-\frac{\alpha^2}{x^2}} \qquad (12)$$

While the CDF[d] has form:

$$F^{d}(x,\delta,\theta,\alpha,\gamma)=\left(1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}}\right)^{d}$$

By using binomial series expansion, get the form:

$$\left(1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}}\right)^{d}=\sum_{l=0}^{\infty}(-1)^{l}\binom{d}{l}e^{-\delta l\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}}$$

By same steps in CDF we get:

$$F^{d}(x,\delta,\theta,\alpha,\gamma)=Ke^{-q-\frac{\alpha^2}{x^2}} \qquad (13)$$

where

$K=\sum_{l=u=v=s=q=0}^{\infty}\frac{(-1)^{l+v+u(1+l\theta)+s+q}}{u!}\binom{d}{l}\binom{v+2u\theta l}{s}\binom{s\gamma}{q}\delta^{u}l^{u}D_{u\theta l,v}$

, and $D_{u\theta l,v}=v^{-1}\sum_{m=1}^{j}\frac{m(u\theta l+1)-v}{m+1}D_{u\theta l,v-m}$, $j\geq 1, D_{u\theta l,0}=1$

To expand the pdf of HWIBX distribution it is done and has following form:

$$\begin{aligned}f(x,\delta,\theta,\alpha,\gamma)&=\Theta 2\alpha^{2}\gamma x^{-3}e^{-\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(h+s)-1}\\&-\mathrm{B}\,2\alpha^{2}\gamma x^{-3}e^{-(w+\gamma+1)\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(s+1)-1}\end{aligned} \qquad (14)$$

where

$\Theta=\sum_{i=z=s=h=0}^{\infty}\frac{(-1)^{i(\theta+1)+\theta-1+z+s+h}}{i!}\,\delta^{i+1}D_{\theta(i+1)-1,z}\binom{z}{s}\binom{1}{h}\theta$

$$B = \sum_{i=z=s=w=0}^{\infty} \frac{(-1)^{i(\theta+1)+\theta-1+z+s+w}}{i!} \delta^{i+1} D_{\gamma,w} D_{\theta(i+1)-1,z} \binom{z}{s} \theta$$

$$D_{\gamma,w} = w^{-1} \sum_{C=1}^{l} \frac{c(\gamma+1)-w}{c+1} D_{\gamma,w-c} \;, \gamma \geq 1, D_{\gamma,0} = 1$$

$$D_{\theta(i+1)-1,z} = z^{-1} \sum_{m=1}^{l} \frac{m\big(\theta(i+1)\big)-z}{z+1} D_{\theta(i+1)-1,z-m} \;, z \geq 1, D_{\theta(i+1)-1,0} = 1$$

3.2. *Quantile function of HWIBX distribution*

The quantile function $Q(u)$ is derived from the equation [14-15]:

$$Q(u) = F^{-1}(u)$$

where $Q(u)$ is the quantity function. Then the quantity function of HWIBX distribution has form:

$$x = \sqrt{-\frac{\alpha^2}{\log\left(1-\left[1-\frac{\left(-\frac{\log(1-u)}{\delta}\right)^{\frac{1}{\theta}}}{\left(-\frac{\log(1-u)}{\delta}\right)^{\frac{1}{\theta}} - W_{-1}\left(-\left(-\frac{\log(1-u)}{\delta}\right)^{\frac{1}{\theta}} \exp\left[\left(-\frac{\log(1-u)}{\delta}\right)^{\frac{1}{\theta}}\right]\right)}\right]^{\frac{1}{\gamma}}\right)}} \quad (15)$$

where $W_{(.)}$ is Lombard function

The following table represents the quantile function values for different values of parameters

Table.1 explains the quantiles for selected parameter values of the HWIBX distribution.

	$(\delta, \theta, \alpha, \gamma)$				
$\boldsymbol{u}$	(1.3,0.7,0.2,0.4)	(1.4,0.3,0.7,0.3)	(0.5,0.8,0.3,0.7)	(1.7,3,0.8,0.3)	(1.6,0.3,0.1,0.2)
0.1	0.1957	0.3959	0.3355	2.7336	0.0582
0.2	0.2527	0.5048	0.4488	3.3793	0.0754
0.3	0.3151	0.6340	0.5842	3.9337	0.0965
0.4	0.3927	0.8157	0.7716	4.4723	0.1272
0.5	0.4989	1.1144	1.0652	5.0343	0.1797
0.6	0.6610	1.7215	1.5997	5.6568	0.2910
0.7	0.9484	3.5055	2.8097	6.3949	0.6331
0.8	1.6086	16.0204	6.7854	7.3610	3.0569
0.9	4.3913	4632.5812	37.3498	8.9023	589.5359

3.3. *Moments*

The r^{th} moment of HWIBX distribution can be finding from relationship [16]:

$$\mu_r = E(x^r)_{HWIBX} = \int_0^{\infty} x^r f(x, \delta, \theta, \alpha, \gamma) dx$$

From equation (16) then:

$$\mu_r = E(x^r)_{HWIBX} = \int_0^{\infty} x^r \left[\Theta 2\alpha^2\gamma x^{-3} e^{-\frac{\alpha^2}{x^2}} \left(1 - e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(h+s)-1} - B 2\alpha^2\gamma x^{-3} e^{-(w+\gamma+1)\frac{\alpha^2}{x^2}} \left(1 - e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(s+1)-1}\right] dx$$

$$= \Theta \int_0^{\infty} 2\alpha^2\gamma x^{r-3} e^{-\frac{\alpha^2}{x^2}} \left(1 - e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(h+s)-1} dx - B \int_0^{\infty} 2\alpha^2\gamma x^{r-3} e^{-(w+\gamma+1)\frac{\alpha^2}{x^2}} \left(1 - e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(s+1)-1} dx$$

Using the change of variable, $t = \frac{1}{1-e^{-\frac{\alpha^2}{x^2}}}, \; 0 < t < 1$, we obtain:

$$\mu_r = \Theta \int_0^1 (\alpha^2\gamma)^{t^{(h+s)-1}} \left(-\left[\ln\left(\frac{1}{t}-1\right)\right]^{\frac{1}{2}} + 1 \right)^r dt - \mathrm{B} \int_0^1 (\alpha^2\gamma)^{t^{\gamma(s+1)-1}} \left(-\left[\ln\left(\frac{1}{t}-1\right)\right]^{\frac{1}{2}} + (w+\gamma+1) \right)^r dt$$

$$\mu_r = \Theta\alpha^2\gamma \sum_{i=0}^{r} \binom{r}{i} (-1)^r \int_0^1 (\alpha^2\gamma)^{t^{(h+s)-1}} \left[\ln\left(\frac{1}{t}-1\right)\right]^{\frac{i}{2}} dt$$

$$- \mathrm{B}\alpha^2\gamma \sum_{j=0}^{r} \binom{r}{j} (-1)^r (w+\gamma+1)^{r-j} \int_0^1 (\alpha^2\gamma)^{t^{\gamma(s+1)-1}} \left[\ln\left(\frac{1}{t}-1\right)\right]^{\frac{j}{2}} dt$$

Now using $\frac{1}{t} - 1 = e^u, 0 < u < \infty$, we obtain

$$\mu_r = \Theta\alpha^2\gamma \sum_{i=0}^{r} \binom{r}{i} (-1)^r \int_0^\infty (e^u+1)^{(h+r)-1} u^{\frac{i}{2}} e^u du$$

$$- \mathrm{B}\alpha^2\gamma \sum_{j=0}^{r} \binom{r}{j} (-1)^r (w+\gamma+1)^{r-j} \int_0^\infty (e^u+1)^{\gamma(r+1)-1} u^{\frac{j}{2}} e^u du$$

$$\mu_r = \Theta\alpha^2\gamma \sum_{i=0}^{r} \binom{r}{i} (-1)^r E_q\big(g(X)\big) - \mathrm{B}\alpha^2\gamma \sum_{j=0}^{r} \binom{r}{j} (-1)^r (w+\gamma+1)^{r-j} E_q\big(k(X)\big) \quad (16)$$

where $E_q(*)$ is denoted to expectation for $X \sim q$, and q is standard exponential distribution and $g(X) = (e^x+1)^{(h+r)-1} x^{\frac{i}{2}} e^{2x}$, $k(X) = (e^x+1)^{\gamma(r+1)-1} x^{\frac{j}{2}} e^{2x}$

The importance sampling approach yields an estimate of μ_r, which is referred to as the importance sampling estimate.

$$\hat{\mu}_{r,q} = \Theta\alpha^2\gamma \sum_{i=0}^{r} \binom{r}{i} (-1)^r \left(\frac{1}{m} \sum_{a=0}^{m} g(X_a) \right) - \mathrm{B}\alpha^2\gamma \sum_{j=0}^{r} \binom{r}{j} (-1)^r (w+\gamma+1)^{r-j} \left(\frac{1}{m} \sum_{a=0}^{m} k(X_a) \right) \quad (17)$$

Mean of $\hat{\mu}_{r,q}$ are given by:

$$E(\hat{\mu}_{r,q}) = \mu_r \quad (18)$$

The variance $\hat{\mu}_{r,q}$ are given by:

$$var(\hat{\mu}_{r,q}) = \Theta\alpha^4\gamma^2 \sum_{i=0}^{r} \left[\binom{r}{i}\right]^2 var\left(\hat{E}_q\big(g(X)\big)\right)$$

$$+ \mathrm{B}\alpha^4\gamma^2 \sum_{j=0}^{r} \left[\binom{r}{j} (w+\gamma+1)^{r-j}\right]^2 var\left(\hat{E}_q\big(k(X)\big)\right) \quad (21)$$

$$-2cov\left(\Theta\alpha^2\gamma \sum_{i=0}^{r} \binom{r}{i} (-1)^r E_q\big(g(X)\big), \mathrm{B}\alpha^2\gamma \sum_{j=0}^{r} \binom{r}{j} (-1)^r (w+\gamma+1)^{r-j} E_q\big(k(X)\big) \right)$$

where $var\left(\hat{E}_q\big(g(X)\big)\right) = E_q\left(g(X) - E_q\big(g(X)\big)^2\right)$, and $\left(\hat{E}_q\big(k(X)\big)\right) = E_q\left(k(X) - E_q\big(k(X)\big)^2\right)$

Table 2 represents the values of the first, second, third and fourth moments, in addition to the values of skewness and kurtoses for different values of parameters.

Table 2: Numerical value of $\mu_1, \mu_2, \mu_3, \mu_4, \sigma^2, SK$, and KU of the HWIBX distribution

δ	θ	α	γ	μ_1	μ_2	μ_3	μ_4	σ^2	SK	KU
1.2	1.5	0.3	1.1	0.461712	0.237108	0.137342	0.091479	0.02393	1.189552	1.627152
			1.2	0.43477	0.207469	0.109791	0.065329	0.018444	1.161816	1.517732
	1.3	0.4	1.3	0.552294	0.342878	0.245769	0.076545	0.037849	1.224104	0.651081
			1.4	0.527248	0.308553	0.204608	0.158891	0.030563	1.193789	1.668934
2.2	1.5	0.5	1.5	0.550733	0.316428	0.189761	0.118855	0.013121	1.066094	1.187049
			1.6	0.534644	0.297284	0.171969	0.103534	0.01144	1.060943	1.171495
	1.7	0.6	1.7	0.635514	0.416287	0.28095	0.195292	0.012409	1.04602	1.126932
			1.8	0.619977	0.395436	0.25937	0.174881	0.011064	1.04305	1.118385

The moment generating function (mgf) gives [17]:

$$M_x(y)_{HWIBX} = E(e^{yx}) = \int_{-\infty}^{\infty} e^{yx} f(x,\delta,\theta,\alpha,\gamma)dx$$

Therefore form equation(18) the (mgf) of the HWIBX distribution is given as flowing:

$$M_x(y)_{HWIBX} = \sum\nolimits_{n=0}^{\infty} \frac{\alpha^2\gamma y^n}{n!}\left[\Theta \sum_{i=0}^{r}\binom{r}{i}(-1)^r E_q\big(g(X)\big) - \mathrm{B}\sum_{j=0}^{r}\binom{r}{j}(-1)^r(w+\gamma+1)^{r-j}E_q\big(k(X)\big)\right] \quad (19)$$

3.4. Probability Weighted Moments

The Probability Weighted Moments of HWIBX distribution can be founding using the following equation [17]:

$$\tau_{q,s} = E\big(x^q F^s(X)\big) = \int_{-\infty}^{\infty} x^q f(x) F^s(x) dx$$

Substituting $F^s(X)$, and $f(x)$ from equation (15) and (16) into the previous equation we get:

$$\tau_{p,s} = \int_{-\infty}^{\infty} x^p\left[\Theta 2\alpha^2\gamma x^{-3} e^{-\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(h+s)-1} - \mathrm{B}\,2\alpha^2\gamma x^{-3} e^{-(w+\gamma+1)\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(s+1)-1}\right] K e^{-p-\frac{\alpha^2}{x^2}} dx$$

$$\tau_{q,s} = \int_{0}^{\infty} \Theta K 2\alpha^2\gamma x^{p-3} e^{-(p+1)\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(h+s)-1} - \mathrm{B}K2\alpha^2\gamma x^{p-3} e^{-(w+\gamma+1+p)\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(s+1)-1} dx$$

In the same way as proving the moment function of the distribution, we obtain the function in the form:

$$\tau_{p,s} = \Theta K\alpha^2\gamma \sum_{i=0}^{p}\binom{p}{i}(-1)^p(p+1)^{p-i}E_q\big(g(X)\big) - \mathrm{B}K\alpha^2\gamma\sum_{j=0}^{p}\binom{p}{j}(-1)^p(w+\gamma+p+1)^{p-j}E_q\big(k(X)\big) \quad (20)$$

3.5. Rényi Entropy

The Rényi entropy for the of HWIBX distribution can be founding using the following equation [12]:

$$I_R(c)_{OLE} = \frac{1}{1-c} log \int_0^{\infty} f(x)^c dx \qquad (21)$$

Then from equation (16):

f

$$(x,\delta,\theta,\alpha,\gamma) = \sum_{i=z=s=h=0}^{\infty} \frac{(-1)^{i(\theta+1)+\theta-1+z+s+h}}{i!} \delta^{i+1} D_{\theta(i+1)-1,z} \binom{z}{s}\binom{1}{h} \theta 2\alpha^2\gamma x^{-3} e^{-\frac{\alpha^2}{x^2}} \left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(h+s)-1} - \sum_{i=z=s=w=0}^{\infty} \frac{(-1)^{i(\theta+1)+\theta-1+z+s+w}}{i!} \delta^{i+1} D_{\gamma,w} D_{\theta(i+1)-1,z} \binom{z}{s} \theta 2\alpha^2\gamma x^{-3} e^{-(w+\gamma+1)\frac{\alpha^2}{x^2}} \left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(s+1)-1}$$

Using the expansion of the above equation as follows:

$$\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(h+s)-1} = \sum_{p=0}^{\infty}(-1)^p \binom{\gamma(h+s)-1}{p} e^{-p\frac{\alpha^2}{x^2}}$$

$$\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma(s+1)-1} = \sum_{q=0}^{\infty}(-1)^q \binom{\gamma(s+1)-1}{q} e^{-q\frac{\alpha^2}{x^2}}$$

f

$$(x,\delta,\theta,\alpha,\gamma) = \sum_{i=z=s=h=p=0}^{\infty} \frac{(-1)^{i(\theta+1)+\theta-1+z+s+h+p}}{i!} \delta^{i+1} D_{\theta(i+1)-1,z} \binom{z}{s}\binom{1}{h}\binom{\gamma(h+s)-1}{p} \theta 2\alpha^2\gamma x^{-3} e^{-(p+1)\frac{\alpha^2}{x^2}} - \sum_{i=z=s=w=q=0}^{\infty} \frac{(-1)^{i(\theta+1)+\theta-1+z+s+w+q}}{i!} \delta^{i+1} D_{\gamma,w} D_{\theta(i+1)-1,z} \binom{z}{s}\binom{\gamma(s+1)-1}{q} \theta 2\alpha^2\gamma x^{-3} e^{-(w+\gamma+q+1)\frac{\alpha^2}{x^2}}$$

Then

$$f(x,\delta,\theta,\alpha,\gamma) = Ax^{-3}e^{-(p+1)\frac{\alpha^2}{x^2}} - Hx^{-3}e^{-(w+\gamma+q+1)\frac{\alpha^2}{x^2}} \qquad (22)$$

Where

$$A = \sum_{i=z=s=h=p=0}^{\infty} \frac{(-1)^{i(\theta+1)+\theta-1+z+s+h+p}}{i!} \delta^{i+1} D_{\theta(i+1)-1,z} \binom{z}{s}\binom{1}{h}\binom{\gamma(h+s)-1}{p} \theta 2\alpha^2\gamma$$

And $B = \sum_{i=z=s=w=q=0}^{\infty} \frac{(-1)^{i(\theta+1)+\theta-1+z+s+w+q}}{i!} \delta^{i+1} D_{\gamma,w} D_{\theta(i+1)-1,z} \binom{z}{s}\binom{\gamma(s+1)-1}{q} \theta 2\alpha^2\gamma$

Substituting equation (25) into (24) we get:

$$I_R(c)_{HWIBX} = \frac{1}{1-c} log \int_0^{\infty} \left(Ax^{-3}e^{-(p+1)\frac{\alpha^2}{x^2}} - Hx^{-3}e^{-(w+\gamma+q+1)\frac{\alpha^2}{x^2}}\right)^c dx$$

Using binomial series expansion, we get:

$$I_R(c)_{HWIBX} = \frac{1}{1-c} \log \left[\sum_{k=0}^{c}(-1)^k \binom{c}{k} A.H \int_0^{\infty} x^{-3c} e^{-[k(p-w-\gamma-q)+c(w+\gamma+q-1)]\frac{\alpha^2}{x^2}} dx\right]$$

$$I_R(c)_{HWIBX} = \frac{1}{1-c} \log \left[\sum_{k=0}^{c} \frac{(-1)^{k+1}\binom{c}{k} A.H\Gamma\left(\frac{3c-1}{2}\right)}{2\alpha^{3c+1}[k(p-w-\gamma-q)+c(w+\gamma+q-1)]^{\frac{3c-1}{2}}}\right] \qquad (23)$$

3.6. Order statistics

The pdf of the $j-th$ order statistic for a random sample of size n from a distribution function $F_{HWE}(x,w,\gamma,b)$ and an associated pdf $f_{HWE}(x,w,\gamma,b)$ is given by [18]:

The probability density function (pdf) of the $j-th$ order statistic for a random sample of size n from a CDF and its related pdf is provided in equation [18].

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} [F_{HWE}(x,w,\gamma,b)]^{j+r-1} f_{HWE}(x,w,\gamma,b) \qquad (24)$$

The probability density function (pdf) for the $j-th$ order statistics of a random sample of size n from the HWIBX distribution can be expressed as:

$$f_{j:n}(x)=\sum_{r=0}^{n-j}k(-1)^r\binom{n-j}{r}\left[1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}}\right]^{j+r-1}$$

$$\times\,\delta\theta\,2\alpha^2\gamma x^{-3}e^{-\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma-1}\left[\frac{1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}}{\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}}-\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right] \quad (25)$$

$$\times\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta-1}e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}}$$

To obtain the order statistics of the minimum, we substitute $j=1$ into equation (28) to find $f_{j:n}(x)$. Conversely, to obtain the order statistics of the maximum, we substitute $j=n$.

4. Estimation

In this section, it is explained the results of research and at the same time is given the comprehensive discussion. Results can be presented in figures, graphs, tables and others that make the reader understand easily [2, 5]. The discussion can be made in several sub-chapters.

The parameters of the HWIBX distribution are determined using the maximum likelihood estimation approach. The log-likelihood function for a random sample $x_1,x_2,\ldots,x_n$ is obtained. The distribution adheres to the probability density function (pdf) of the HWIBX distribution [19].

$$L(\Phi,x)=\prod_{i=1}^{n}f(x,\delta,\theta,\alpha,\gamma)$$

4.1 Maximum Likelihood Estimation (MLE)

$$L(\Phi,x_i)=\prod_{i=1}^{n}\delta\theta\,2\alpha^2\gamma x^{-3}e^{-\frac{\alpha^2}{x^2}}\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma-1}\left[\frac{1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}}{\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}}-\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]$$

$$\times\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta-1}e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]^{\theta}}$$

The log-likelihood function L is obtained as:

$$L=nlog(\delta)+nlog(\theta)+nlog2+2nlog(\alpha)+nlog(\gamma)-\sum_{i=1}^{n}\frac{\alpha^2}{x_i}$$

$$+(\gamma-1)\sum_{i=1}^{n}\log\left[1-e^{-\frac{\alpha^2}{x^2}}\right]+\sum_{i=1}^{n}\log\left[\frac{1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}}{\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}}-\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right] \quad (26)$$

$$+(\theta-1)\sum_{i=1}^{n}\log\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]$$

$$-\delta\sum_{i=1}^{n}\left[\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right]\right]^{\theta}$$

To obtain the solution, we calculate the partial derivatives of the distribution parameters and solve the non-linear equations for $\frac{\partial L}{\partial \delta}=\frac{\partial L}{\partial \theta}=\frac{\partial L}{\partial \alpha}=\frac{\partial L}{\partial \gamma}=0$ provides the maximum likelihood estimates of the parameters $\delta, \theta, \alpha,$ and γ, in that order. Analysis did not yield a solution. The only way to locate it was using numerical methods. These methods utilize software like as R, MAPLE, SAS, and other similar programs.

4.2 Ordinary Least Squares Estimation (OLSE)

The OLSE can be founding by using equation [20]:

$$\varphi(\delta,\theta,\alpha,\gamma)=\sum_{i=1}^{n}\left[F(x_i)-\frac{1}{n+1}\right]^2 \tag{27}$$

$$\varphi(\delta,\theta,\alpha,\gamma)=\sum_{i=1}^{n}\left[1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}-\frac{1}{n+1}\right]^2 \tag{28}$$

By partially deriving the above equation for the $\delta, \theta, \alpha, \gamma$ parameters

$$\frac{\partial \varphi}{\partial \delta}=2\sum_{i=1}^{n}\left[1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}-\frac{1}{n+1}\right]\frac{\partial\left(-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}\right)}{\partial \delta} \tag{29}$$

$$\frac{\partial \varphi}{\partial \theta}=2\sum_{i=1}^{n}\left[1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}-\frac{1}{n+1}\right]\frac{\partial\left(-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}\right)}{\partial \theta} \tag{30}$$

$$\frac{\partial \varphi}{\partial \alpha}=2\sum_{i=1}^{n}\left[1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}-\frac{1}{n+1}\right]\frac{\partial\left(-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}\right)}{\partial \alpha} \tag{31}$$

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$$\frac{\partial\varphi}{\partial\gamma}=2\sum_{i=1}^{n}\left[1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}-\frac{1}{n+1}\right]\frac{\partial\left(-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}\right)}{\partial\gamma} \tag{32}$$

By setting the previous equations equal to zero, we obtain the OLSE:

$$\frac{\partial L}{\partial\delta}=\frac{\partial L}{\partial\theta}=\frac{\partial L}{\partial\alpha}=\frac{\partial L}{\partial\gamma}=0$$

We note that the equations are equal to zero. It is clear that it is not possible to obtain the closed form of the above equations and it is difficult to solve them manually. Therefore, it is necessary to use computer programs or numerical methods to find an estimate of these parameters.

4.3 Weighted Least Squares Estimators (WLSE)

The weighted least squares estimators can be obtained by the equation [21-22]:

$$\omega(\delta,\theta,\alpha,\gamma)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i)-\frac{i}{n+1}\right]^2 \tag{33}$$

$$\omega(\delta,\theta,\alpha,\gamma)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-e^{-\delta\left[-\left(1-\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right)\log\left(1-e^{-\frac{\alpha^2}{x_i^2}}\right)^{\gamma}\right]^{\theta}}-\frac{i}{n+1}\right]^2 \tag{34}$$

In the same way as MLE and OLSE, the above equation is derived for the HWIBX distribution parameters and equalized to zero to obtain parameter estimates using the WLSE method.

5. Simulation Study

The performance of MLE, OLSE, and WLSE, for the HWIBX distribution is evaluated through a Monte Carlo simulation study using the R package. The sample sizes considered in the study are n = 50,100, 150, and 200. We generate N = 1000 samples for the true parameter values listed in Tables 2 and 3. The resulting MLEs for the model parameters are averaged to obtain the mean values, and the corresponding bias and root mean squared errors (RMSEs) are calculated. The bias and RMSE for a specific estimated parameter, denoted as $\hat{\lambda}$, are given by:

The effectiveness of MLE, OLSE, and WLSE for the HWIBX distribution is assessed using the R program in a Monte Carlo simulation exercise. The study takes into account sample sizes of n = 50, 100, 150, and 200. For the genuine parameter values given in Tables 2 and 3, we produce N = 1000 samples. After averaging the resulting MLEs for the model parameters, the mean values, associated bias, and root mean squared errors (RMSEs) are computed. For a given estimated parameter, $\hat{\lambda}$, the bias and RMSE are provided by [5]:

$$Abias(\hat{\lambda}) = \frac{\sum_{i=1}^{N} \hat{\lambda}_i}{N} - \lambda, \quad and$$

$$RMSE(\hat{\lambda}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\lambda}_i - \lambda)^2}{N}}.$$

The consistency of all estimators is evident in the results presented in Tables.3. As the sample size increases, the average parameter estimations converge towards the true parameter values. Furthermore, the mean square errors (MSEs) exhibit a decrease in magnitude as the sample size grows.

Table 3 : Monte Carlo simulations conducted for the HWIBX distribution

$\boldsymbol{\delta = 2}$, $\boldsymbol{\theta = 3.3}$, $\boldsymbol{\alpha = 2.3}$, $\boldsymbol{\gamma = 1.4}$					
n	**Est.**	**Est.par.**	**MLE**	**OLS**	**WLSE**
50	MEAN	$\hat{\delta}$	3.533227	2.795115	3.241086
		$\hat{\theta}$	3.612634	2.9727103	2.9275136
		$\hat{\alpha}$	2.516654	2.5720373	2.6204994
		$\hat{\gamma}$	1.7711938	1.6469178	1.7086167
	MSE	$\hat{\delta}$	14.439717	3.770152	10.738688
		$\hat{\theta}$	3.661410	0.9655770	1.4188633
		$\hat{\alpha}$	1.318702	0.5470467	0.8065982
		$\hat{\gamma}$	1.7662692	0.5402421	0.7594095
	RMSE	$\hat{\delta}$	3.799963	1.941688	3.276994
		$\hat{\theta}$	1.913481	0.9826378	1.1911605
		$\hat{\alpha}$	1.148347	0.7396260	0.8981081
		$\hat{\gamma}$	1.3290106	0.7350117	0.8714410
	BAIS	$\hat{\delta}$	1.533227	0.795115	1.241086
		$\hat{\theta}$	0.312634	0.3272897	0.3724864
		$\hat{\alpha}$	0.216654	0.2720373	0.3204994
		$\hat{\gamma}$	0.3711938	0.2469178	0.3086167
100	MEAN	$\hat{\delta}$	3.650430	2.4418134	2.6028521
		$\hat{\theta}$	3.5481211	3.0476143	3.20347344
		$\hat{\alpha}$	2.5129338	2.4404234	2.4138491
		$\hat{\gamma}$	1.6964338	1.5203395	1.48811284
	MSE	$\hat{\delta}$	14.326747	1.0612344	2.7047302
		$\hat{\theta}$	3.6403682	0.6856245	0.86091642
		$\hat{\alpha}$	1.2371966	0.2396901	0.3389660
		$\hat{\gamma}$	1.1895113	0.2191615	0.21353379
	RMSE	$\hat{\delta}$	3.785069	1.0301623	1.6446064
		$\hat{\theta}$	1.9079749	0.8280244	0.92785582
		$\hat{\alpha}$	1.1122934	0.4895815	0.5822078
		$\hat{\gamma}$	1.0906472	0.4681469	0.46209716
	BAIS	$\hat{\delta}$	1.650430	0.4418134	0.6028521
		$\hat{\theta}$	0.2481211	0.2523857	0.09652656
		$\hat{\alpha}$	0.2129338	0.1404234	0.1138491
		$\hat{\gamma}$	0.2964338	0.1203395	0.08811284
150	MEAN	$\hat{\delta}$	3.139756	2.5282068	2.5626724
		$\hat{\theta}$	3.6611341	3.0712860	3.1881323
		$\hat{\alpha}$	2.38055727	2.4748243	2.4385233
		$\hat{\gamma}$	1.5471682	1.5440444	1.5242848
	MSE	$\hat{\delta}$	9.598982	1.6093101	1.9338932
		$\hat{\theta}$	2.8559607	0.5478837	0.7727520

		$\hat{\alpha}$	0.89420271	0.2328435	0.3205291
		$\hat{\gamma}$	0.7133231	0.1869086	0.2533374
	RMSE	$\hat{\delta}$	3.098222	1.2685859	1.3906449
		$\hat{\theta}$	1.6899588	0.7401916	0.8790631
		$\hat{\alpha}$	0.94562292	0.4825386	0.5661529
		$\hat{\gamma}$	0.8445846	0.4323293	0.5033264
	BAIS	$\hat{\delta}$	1.139756	0.5282068	0.5626724
		$\hat{\theta}$	0.3611341	0.2287140	0.1118677
		$\hat{\alpha}$	0.08055727	0.1748243	0.1385233
		$\hat{\gamma}$	0.1471682	0.1440444	0.1242848
200	MEAN	$\hat{\delta}$	3.076449	2.3501330	2.6860643
		$\hat{\theta}$	3.5247064	3.0876492	3.0722768
		$\hat{\alpha}$	2.4324132	2.4419260	2.4956470
		$\hat{\gamma}$	1.6021240	1.5167150	1.5625423
	MSE	$\hat{\delta}$	8.437537	0.8999127	2.4676514
		$\hat{\theta}$	2.5211431	0.5051947	0.7482284
		$\hat{\alpha}$	0.8581232	0.1625232	0.3194223
		$\hat{\gamma}$	0.8126238	0.1072655	0.2267768
	RMSE	$\hat{\delta}$	2.904744	0.9486373	1.5708760
		$\hat{\theta}$	1.5878108	0.7107705	0.8650020
		$\hat{\alpha}$	0.9263494	0.4031416	0.5651746
		$\hat{\gamma}$	0.9014565	0.3275142	0.4762109
	BAIS	$\hat{\delta}$	1.076449	0.3501330	0.6860643
		$\hat{\theta}$	0.2247064	0.2123508	0.2277232
		$\hat{\alpha}$	0.1324132	0.1419260	0.1956470
		$\hat{\gamma}$	0.2021240	0.1167150	0.1625423

$\boldsymbol{\delta = 4}$, $\boldsymbol{\theta = 6.3}$, $\boldsymbol{\alpha = 3.3}$, $\boldsymbol{\gamma = 2.4}$

n	Est.	Est.par.	MLE	LSE	WLSE
50	MEAN	$\hat{\delta}$	4.6565847	4.1445026	4.3621907
		$\hat{\theta}$	6.6281004	6.6056603	6.5624628
		$\hat{\alpha}$	3.34651455	3.31549803	3.32118247
		$\hat{\gamma}$	2.48414161	2.44476330	2.43834679
	MSE	$\hat{\delta}$	8.0844127	1.7224036	2.7480082
		$\hat{\theta}$	1.2694912	1.6773742	1.5190723
		$\hat{\alpha}$	0.04995528	0.01158361	0.02935605
		$\hat{\gamma}$	0.14586097	0.03960609	0.06066491
	RMSE	$\hat{\delta}$	2.8433102	1.3124037	1.6577117
		$\hat{\theta}$	1.1267170	1.2951348	1.2325065
		$\hat{\alpha}$	0.22350677	0.10762720	0.17133608
		$\hat{\gamma}$	0.38191749	0.19901279	0.24630247
	BAIS	$\hat{\delta}$	0.6565847	0.1445026	0.3621907
		$\hat{\theta}$	0.3281004	0.3056603	0.2624628
		$\hat{\alpha}$	0.04651455	0.01549803	0.02118247
		$\hat{\gamma}$	0.08414161	0.04476330	0.03834679
100	MEAN	$\hat{\delta}$	4.8177497	4.5478644	4.3680454
		$\hat{\theta}$	6.39391462	6.33759859	6.290371653
		$\hat{\alpha}$	3.31701680	3.317310301	3.309569736

		$\hat{\gamma}$	2.43268957	2.409427233	2.409829413
	MSE	$\hat{\delta}$	21.7372154	2.0386559	1.2314025
		$\hat{\theta}$	0.70683093	0.51471140	0.594872857
		$\hat{\alpha}$	0.07130379	0.006441215	0.013439729
		$\hat{\gamma}$	0.21448073	0.016204475	0.038181128
	RMSE	$\hat{\delta}$	4.6623187	1.4278151	1.1096858
		$\hat{\theta}$	0.84073238	0.71743390	0.771280012
		$\hat{\alpha}$	0.26702770	0.080257179	0.115929846
		$\hat{\gamma}$	0.46312065	0.127296801	0.195399919
	BAIS	$\hat{\delta}$	0.8177497	0.5478644	0.3680454
		$\hat{\theta}$	0.09391462	0.03759859	0.009628347
		$\hat{\alpha}$	0.01701680	0.017310301	0.009569736
		$\hat{\gamma}$	0.03268957	0.009427233	0.009829413
150	MEAN	$\hat{\delta}$	4.571714	4.2293653	4.2435405
		$\hat{\theta}$	6.37229209	6.4072992	6.4148463
		$\hat{\alpha}$	3.31982259	3.293922589	3.28940356
		$\hat{\gamma}$	2.42394264	2.391758105	2.390993335
	MSE	$\hat{\delta}$	4.194430	0.9489448	1.3472771
		$\hat{\theta}$	0.44881299	0.4137572	0.3745817
		$\hat{\alpha}$	0.03551712	0.007691098	0.01719630
		$\hat{\gamma}$	0.07823679	0.022684789	0.041235885
	RMSE	$\hat{\delta}$	2.048031	0.9741380	1.1607227
		$\hat{\theta}$	0.66993507	0.6432396	0.6120308
		$\hat{\alpha}$	0.18845987	0.087698908	0.13113465
		$\hat{\gamma}$	0.27970840	0.150614703	0.203066208
	BAIS	$\hat{\delta}$	0.571714	0.2293653	0.2435405
		$\hat{\theta}$	0.07229209	0.1072992	0.1148463
		$\hat{\alpha}$	0.01982259	0.006077411	0.01059644
		$\hat{\gamma}$	0.02394264	0.008241895	0.009006665
200	MEAN	$\hat{\delta}$	4.7839037	4.2936929	4.2820777
		$\hat{\theta}$	6.28888455	6.39346802	6.4018664
		$\hat{\alpha}$	3.32818622	3.28054853	3.297273598
		$\hat{\gamma}$	2.43676011	2.36457127	2.394805195
	MSE	$\hat{\delta}$	14.7660078	1.4372242	0.9168381
		$\hat{\theta}$	0.40519123	0.35144301	0.2243305
		$\hat{\alpha}$	0.04325338	0.01292535	0.010174917
		$\hat{\gamma}$	0.11181533	0.02772834	0.021711553
	RMSE	$\hat{\delta}$	3.8426564	1.1988429	0.9575166
		$\hat{\theta}$	0.63654633	0.59282629	0.4736355
		$\hat{\alpha}$	0.20797447	0.11368971	0.100870793
		$\hat{\gamma}$	0.33438799	0.16651829	0.147348407
	BAIS	$\hat{\delta}$	0.7839037	0.2936929	0.2820777
		$\hat{\theta}$	0.01111545	0.09346802	0.1018664
		$\hat{\alpha}$	0.02818622	0.01945147	0.002726402
		$\hat{\gamma}$	0.03676011	0.03542873	0.005194805

$\boldsymbol{\delta = 7}$, $\boldsymbol{\theta = 5.2}$, $\boldsymbol{\alpha = 2.9}$, $\boldsymbol{\gamma = 1.7}$

n	**Est.**	**Est.par.**	**MLE**	**LSE**	**WLSE**

50	MEAN	$\hat{\delta}$	15.584191	7.05402296	7.8600543
		$\hat{\theta}$	4.7812083	5.24369301	5.11998217
		$\hat{\alpha}$	3.3305942	2.90397616	2.94269831
		$\hat{\gamma}$	2.2199612	1.71074600	1.74675376
	MSE	$\hat{\delta}$	441.523528	4.84737169	17.8549030
		$\hat{\theta}$	3.2044866	0.81397292	0.80687806
		$\hat{\alpha}$	1.0311903	0.02340559	0.11261165
		$\hat{\gamma}$	1.3229076	0.03102838	0.13286556
	RMSE	$\hat{\delta}$	21.012461	2.20167475	4.2255062
		$\hat{\theta}$	1.7901080	0.90220448	0.89826391
		$\hat{\alpha}$	1.0154754	0.15298887	0.33557660
		$\hat{\gamma}$	1.1501772	0.17614875	0.36450728
	BAIS	$\hat{\delta}$	8.584191	0.05402296	0.8600543
		$\hat{\theta}$	0.4187917	0.04369301	0.08001783
		$\hat{\alpha}$	0.4305942	0.00397616	0.04269831
		$\hat{\gamma}$	0.5199612	0.01074600	0.04675376
100	MEAN	$\hat{\delta}$	13.646749	7.8754887	7.966437
		$\hat{\theta}$	4.7553206	5.16183201	5.16968444
		$\hat{\alpha}$	3.2346413	2.95224905	2.91849311
		$\hat{\gamma}$	2.0489681	1.74369812	1.70458840
	MSE	$\hat{\delta}$	310.075351	6.3685493	12.035125
		$\hat{\theta}$	2.1011739	0.59129304	0.56648966
		$\hat{\alpha}$	0.6014580	0.02905953	0.05387000
		$\hat{\gamma}$	0.6746226	0.02689644	0.04652206
	RMSE	$\hat{\delta}$	17.608957	2.5235985	3.469168
		$\hat{\theta}$	1.4495426	0.76895581	0.75265507
		$\hat{\alpha}$	0.7755372	0.17046856	0.23209911
		$\hat{\gamma}$	0.8213541	0.16400135	0.21568974
	BAIS	$\hat{\delta}$	6.646749	0.8754887	0.966437
		$\hat{\theta}$	0.4446794	0.03816799	0.03031556
		$\hat{\alpha}$	0.3346413	0.05224905	0.01849311
		$\hat{\gamma}$	0.3489681	0.04369812	0.00458840
150	MEAN	$\hat{\delta}$	11.767101	7.356624	7.8200919
		$\hat{\theta}$	4.8017043	5.27382481	5.0698457
		$\hat{\alpha}$	3.1854227	2.90716845	2.95598433
		$\hat{\gamma}$	2.0295211	1.71109310	1.74951313
	MSE	$\hat{\delta}$	186.533415	5.576696	9.2385139
		$\hat{\theta}$	1.6586243	0.32922031	0.4618655
		$\hat{\alpha}$	0.5102497	0.01216502	0.06217736
		$\hat{\gamma}$	0.6521344	0.00891328	0.05730505
	RMSE	$\hat{\delta}$	13.657724	2.361503	3.0394924
		$\hat{\theta}$	1.2878759	0.57377723	0.6796069
		$\hat{\alpha}$	0.7143177	0.11029515	0.24935388
		$\hat{\gamma}$	0.8075484	0.09441019	0.23938474
	BAIS	$\hat{\delta}$	4.767101	0.356624	0.8200919
		$\hat{\theta}$	0.3982957	0.07382481	0.1301543
		$\hat{\alpha}$	0.2854227	0.00716845	0.05598433
		$\hat{\gamma}$	0.3295211	0.01109310	0.04951313

200	MEAN	$\hat{\delta}$	12.739632	7.5617582	7.317860
		$\hat{\theta}$	4.8784568	5.11472879	5.16880645
		$\hat{\alpha}$	3.1665512	2.91452163	2.91955858
		$\hat{\gamma}$	1.9922837	1.70654087	1.71561552
	MSE	$\hat{\delta}$	216.915430	3.1619837	2.940861
		$\hat{\theta}$	1.7158683	0.23318992	0.29891293
		$\hat{\alpha}$	0.5715423	0.01743704	0.01586472
		$\hat{\gamma}$	0.7025438	0.01896648	0.01421274
	RMSE	$\hat{\delta}$	14.728049	1.7781968	1.714894
		$\hat{\theta}$	1.3099116	0.48289742	0.54672930
		$\hat{\alpha}$	0.7560042	0.13204940	0.12595523
		$\hat{\gamma}$	0.8381789	0.13771886	0.11921721
	BAIS	$\hat{\delta}$	5.739632	0.5617582	0.317860
		$\hat{\theta}$	0.3215432	0.08527121	0.03119355
		$\hat{\alpha}$	0.2665512	0.01452163	0.01955858
		$\hat{\gamma}$	0.2922837	0.00654087	0.01561552

6. Application

In order to showcase the efficacy of the HWIBX distribution in accurately fitting data, we provide a real-world example using two data sets. The objective is to highlight the benefits of HWIBX and the degree to which it aligns with the data. Table 4 presents a comparative analysis of HWIBX and various distributions for the utilized data.

Table 4: CDF functions for comparative distributions

Distribution	CDF
[0,1] Truncated Nadarajah-Haghighi Inverse Burr X ([0,1]NHIBX)	$\frac{1-e^{1-\left(1+\delta\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)\right)^{\theta}}}{1-e^{1-(1-\delta)^{\theta}}}$
Kumaraswamy Inverse Burr X (KuIBX) (New)	$1-\left(1-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)^{\delta}\right)^{\theta}$
Exponential Generalized Inverse Burr X (EGIBX) (New)	$\left(1-\left(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right)^{\delta}\right)^{\theta}$
Log Gamma Inverse Burr X (LGamIBX)	$1-\Gamma\left(-\delta\log\left(\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}\right),\theta\right)$
Beta Inverse Burr X (BeIBX) [19]	$p\alpha^2\gamma(1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma},\delta,\theta)$
Inverse Burr X (IBX) [10]	$1-\left(1-e^{-\frac{\alpha^2}{x^2}}\right)^{\gamma}$

This comparison is conducted using eight metrics, namely the Kolmogorov-Smirnov statistic (KS), the Anderson-Darling statistic (A), the Cramér-von Mises statistic (W), the information criteria HQIC [23-24], BIC, AIC, and CAIC [25], as well as the p-value associated with the KS test. These measures are commonly employed to assess the quality of fit.

Table 5 and Table 6 display the HWIBX distribution that has the lowest values of AIC, AICC, and BIC when compared to the corresponding values of non-overlapping distributions. In addition, the goodness-of-fit statistics A, W, KS tests, and p-value all suggest that the HWIBX distribution is the most suitable fit for both data I and data II.

- **The First Dataset I**

This application focuses on the duration of remission (in months) for a group of 128 patients who have been diagnosed with bladder cancer. This data has undergone recent analysis in numerous studies, including [26]. The dataset contains the following values:
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

vars	n	mean	sd	median	trimmed	mad	Min	max	range	skew	KU	SE
1	128	9.37	10.51	6.39	7.42	5.46	0.08	79.05	78.97	3.25	15.2	0.93

Table 5: Estimates of models for data1

Dist.	-2L	AIC	CAIC	BIC	HQIC
HWIBX	**412.1536**	**832.3073**	**832.6325**	**843.7154**	**836.9424**
[0,1]NHIBX	461.7729	931.5616	931.8868	942.9697	936.1968
KuIBX	415.2403	838.4805	838.8057	849.8886	843.1157
EGIBX	447.3125	902.625	902.9502	914.0331	907.2602
LGamIBX	425.1468	858.2937	858.6189	869.7018	862.9288
BeIBX	424.5922	857.1844	857.5096	868.5925	861.8196
IBX	505.1199	1014.24	1014.336	1019.944	1016.557

Table 6: Evaluate statistical metrics for the data1

Dist.	W	A	K-S	p-value
HWIBX	**0.06721345**	**0.4552835**	**0.05229047**	**0.8752071**
[0,1]NHIBX	0.7652793	4.72932	0.3145878	1.986578e-11
KuIBX	0.1180385	0.8011623	0.06114451	0.724887
EGIBX	0.8020442	4.875024	0.1513152	0.005694233
LGamIBX	38.55569	250.3297	0.9999987	0
BeIBX	0.3241705	2.080104	0.1003474	0.1518128
IBX	2.12294	12.00713	0.3543045	2.209344e-14

Table 7: parameter estimators by MLE for the data1

Dist.	δ	θ	α	γ
HWIBX	**3.260230462**	**4.601609001**	**0.008892513**	**0.082594196**
[0,1]NHIBX	0.08993546	0.99653341	0.29647638	0.61361551
KuIBX	22.545247752	26.350177927	0.005094885	0.135104612
EGIBX	0.16824563	33.86453268	0.02153313	2.11511660
LGamIBX	27.358862809	3.095337355	0.008749932	0.680536116
BeIBX	37.645205597	18.100188640	0.005253935	0.081712914
IBX	-	-	0.3501005	0.1769828

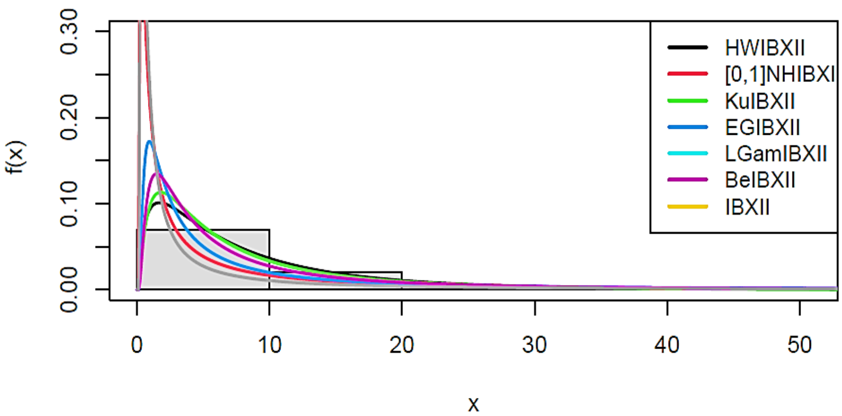

Figure.5 Fitted densities for Data I

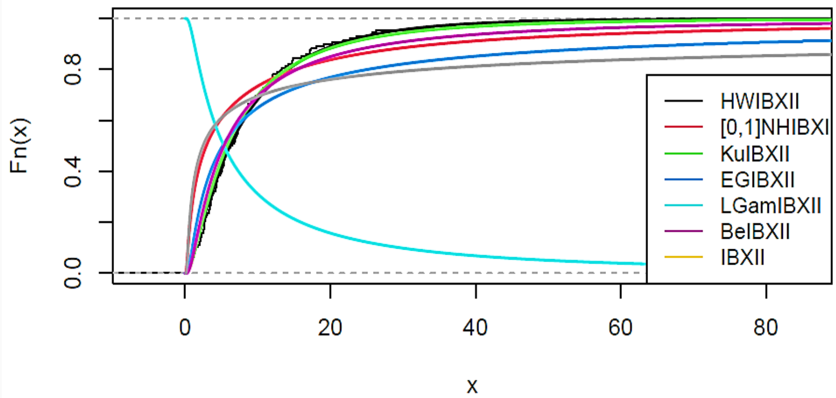

Figure.6 Fitted empirical CDF for Data I

- **The Second Dataset II**

The dataset included in this study consisted of the survival periods, measured in years, of a cohort of 46 patients who received chemotherapy treatment exclusively. The survival times (in years) in this data set were previously reported for easy access [27]:

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507,0.529,0.534,0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

vars	n	Mean	sd	median	trimmed	mad	min	max	range	skew	KU	SE
1	45	1.34	1.25	0.84	1.19	0.95	0.05	4.03	3.99	0.94	-0.45	0.19

Table 8: Estimates of models for data2

Dist.	-2L	AIC	CAIC	BIC	HQIC
HWIBX	**58.2242**	**124.4484**	**125.4484**	**131.6751**	**127.1424**
[0,1]NHIBX	64.62518	137.2504	138.2504	144.477	139.9444
KuIBX	59.77862	127.5572	128.5572	134.7839	130.2513
EGIBX	84.1579	176.3158	177.3158	183.5424	179.0098
LGamIBX	60.84979	129.6996	130.6996	136.9262	132.3936
BeIBX	61.83997	131.6799	132.6799	138.9066	134.374
IBX	72.14705	148.2941	148.5798	151.9074	149.6411

Table 9: Evaluate statistical metrics for the data2

Dist.	W	A	K-S	p-value
HWIBX	**0.05396241**	**0.3980271**	**0.08529341**	**0.871218**
[0,1]NHIBX	0.1329864	0.9634076	0.1924543	0.06197589
KuIBX	0.07235006	0.5343113	0.1026565	0.691647
EGIBX	0.5826623	3.393419	0.3264961	8.996611e-05
LGamIBX	15.18441	89.70491	0.9985948	0
BeIBX	0.1056582	0.7546332	0.1126633	0.5784162

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IBX	0.3059827	1.941572	0.2253925	0.01722965

Table 10: parameter estimators by MLE for the data2

Dist.	δ	θ	α	γ
HWIBX	**7.3109037**	**2.9884617**	**0.0074122**	**0.0864386**
[0,1]NHIBX	14.554068575	2.602689186	0.120772617	0.004792009
KuIBX	9.474044085	9.572313591	0.008266733	0.155861632
EGIBX	2.09102739	0.25800972	0.27219926	0.05589438
LGamIBX	15.817747271	1.109489996	0.006555005	1.485922500
BeIBX	10.97833848	6.27498851	0.01318763	0.12937126
IBX	-	-	0.1344021	0.2715192

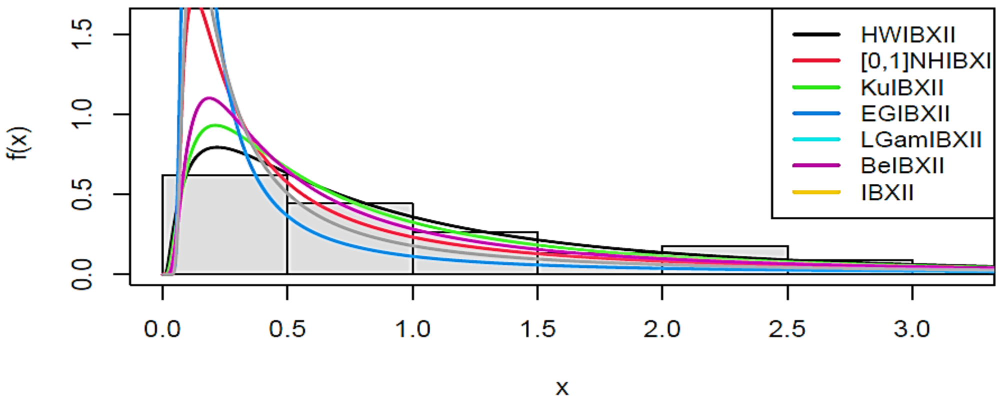

Figure.7 Fitted densities for Data II

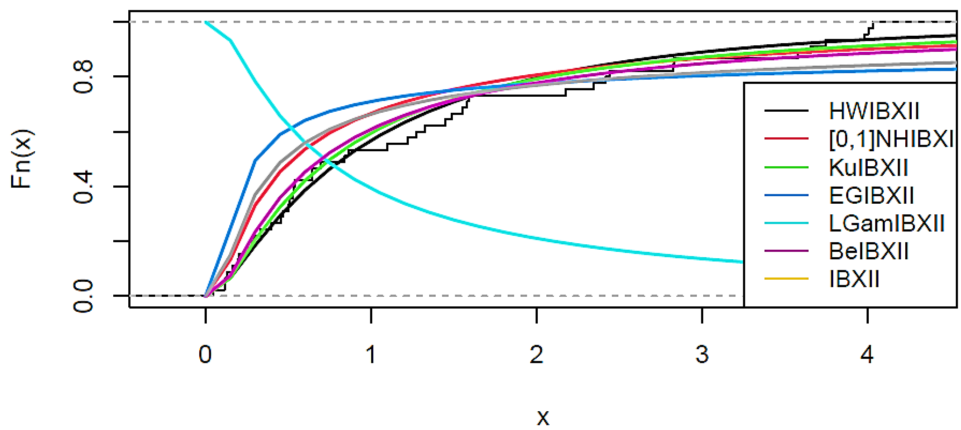

Figure.8 Fitted empirical CDF for Data II

7. Conclusion

The distribution of HWIBX was the focus of this study. The basic distribution functions were found, as well as some mathematical properties of the distribution were proven. Three conventional estimation techniques were also used to derive the unknown parameters of the HWIBX distribution. In terms of the efficiency of estimating simulation values, the OLSE technique performs better than other methods, according to the results presented in the simulation section. Based on the actual data examples, we can see that the HWIBX distribution outperforms all compared distributions in terms of data fit, followed by KuIBX and BeIBX in terms of fit, but the HWIBX distribution still outperforms by a very large amount in terms of results. Finally, we demonstrated that the HWIBX distribution works best when modeling and fitting data from the fields of medicine, economics, industry, etc.

As a proposed future study, the scope of our investigation can be expanded to include application of the proposed model to a variety of accelerated life test scenarios, such as fully static and partially static tests, and perhaps even to the results of progressive load accelerated life tests. This would allow us to better understand how the proposed model can be used to predict the results of these types of tests. This expansion of our focus would require the accumulation of additional information.

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