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Bathtub Curve in Reliability Analysis : A Review

Suhad Ahmed Abdullah [1], **Rabab Abdulrida Saleh**[2]

[1,2]Department of statistics, Collage of Administration and Economics, University of Baghdad, Baghdad, Iraq

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ABSTRACT

The bathtub curve is a popular concept and an important tool in reliability analysis. It describes the pattern or shape of the hazard rate function of products or systems throughout their life cycles over time. It helps in understanding failure modes and identifying critical periods in the systems life cycle in many fields such as engineering, quality management, electronics manufacturing, and other applied fields. This paper reviews the stages of the curve in some detail, along with everything that has been written by researchers on statistical estimation methods for the failure rate at each stage, high lighting the distributions related to this curve. It also addresses how to identify the points of change and transformation in this curve and the extent of its impact on preventive maintenance. The paper concludes with some critical points and bjections to the bathtub curve.

1. Introduction

The bathtub curve was discovered and crystallized as a concept and tool for mathematically and formally describing the behavior of the failure rate of electronic products or systems throughout their life cycle in the 1950s and 1960s, when the electronics and communications industry began to notice that failure rates were not constant, but rather variable and passed through three stages. One of the first description of the bathtub curve was attributed to engineers at the Bell Telephone Laboratories (Bell System Technical Journal 1976). In the field of reliability, The first academic description of this curve was done by researchers Barlow & Prochain in 1965 in their book (The Mathematical Theory of Reliability). The name bathtub curve comes from the curve that resembles the longitudinal section of a bathtub, which in reliability analysis represents the failure rate of one or more assets by plotting their failure states.

Failure rate functions can be classified into three groups.

1- Monotonically increasing or decreasing failure rates

2- Bathtub-shaped or U-shaped failure rates

3-Roller-coaster-shaped failure rates, also called generalized bathtub failure rates.

Therefore, this review aims to clarify the periods included in the bathtub curve, how to identify change points, its impact on preventive maintenance and fuzzy reliability, and finally, to identify the statistical distributions for which the failure rate takes the shape of a bathtub.

2. Theoretical description of the bathtub curve

Glaser (1980) defined that the failure rate function has the shape of a bathtub if there is a time t_0 and it is: (9)

Corresponding author E-mail address: Suhad.a@coadec.uobaghdad.edu.iq
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i. h(t) is decreasing for every t<t_0
ii. h(t) is increasing for every t>t_0
That means

$\acute{h}(t) \acute{>} 0$ ،$h(t) < 0, \acute{h}(t_0\,) = 0$

The second description was given by researcher Mitra in 1995, where he said that the life distribution F is a continuous distribution within the period($0, \infty$]and is a distribution with a failure rate that takes the form of a bathtub if there is t_0 ≥0 such that the failure rate function is not increasing over the period $(0, t_0$]and not decreasing over the period
[t_0, ∞). In such a case, we say that the distribution has a bathtub failure rate function wheret_0 refers to the change point according to the researcher's nomenclature) .
As for the researcher Mi. J in the same year he gave a description of the life distribution F that has the shape of a bathtub if this condition $0 \le t_1 \le t_2 < \infty$ exists, such that the failure rate function is increasing strongly if $0 \le t \le t_1$and is constant if $t_1 \le t \le t_2$, where t_1 and t_2 are the change points, and he considered that the monotonous increasing and decreasing functions are a special case of the bathtub.
This definition was set by the scientists Haupt and Scabe in 1997. The failure rate function takes the form of a bathtub if t_0 exists and
i. $\bar{F}(x|t)$ be increasing in t for

$$0 \le x \le t_0 - \text{t}, 0 \le t < t_0$$

ii. $\bar{F}(x|t)$ be decreasing in t for

$$t_0 \le t < \infty, x \ge 0$$

3-Stages of bathtub curve

The bathtub curve goes through three stages. The stages of the curve were explained by a group of researchers. Products or systems typically experience or go through three main life stages, which represent the three regions of the bathtub curve, which have been described by many researchers, Glaser (1980), George A. Klutke et al (2003), L.C.Mendez-Gonzalez et al. (2022), (Wikipedia,2025) and (Vigneshwar ,2025).

i. Infant mortality region: This stage begins with a high failure rate and then begins to decline rapidly over time when the product is newly introduced or the device or system is just starting to work. This is due to manufacturing problems or design flaws that developed at the beginning of use.

ii. Useful life region :The stable failure rate zone is when the product is stable and continues to operate without failures at a low failure rate and failures are random and not related to physical deterioration.

iii. Wear-out region: In this region or stage, it is noted that the failure rate rises again due to the natural wear and tear of the material parts. This occurs at the end of the product's life cycle, as shown in Figure (1).

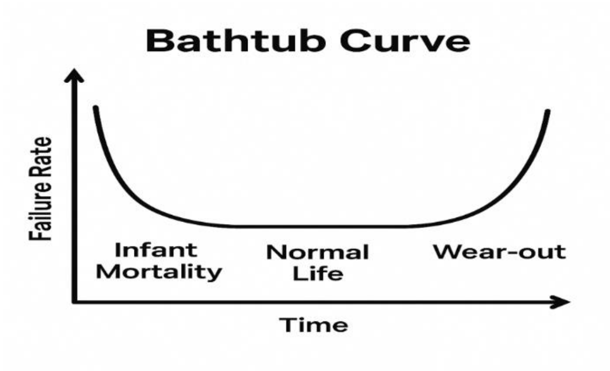

Figure 1. Reliability bathtub failure rate

4- Change points and turning points for the bathtub curve

The change points and turning points of the bathtub curve are defined as the dividing points between the early failure stage, the normal use stage or operational life stage, and the wear stage. Determining these points has practical importance.
Kuocheng & Kwei Tang in (1992) used a mixture of Weibull - Hyper exponential distribution with a shape parameter less than one to describe the infant mortality stage, which is the first stage of the bathtub curve and the natural stages of the product's life. The maximum likelihood method was used to estimate the change point and study the effect

of sample size on the change error and model performance.

M. Xie and others (2004) presented a paper in which they discussed the relationship between the failure rate function and the remaining life function, which is one of the important characteristics in reliability analysis. The focus was on the basic correlations between the two functions and on the difference between the change points, as the change point for the remaining life function is much larger than the change point for the failure rate function, which takes the form of a bathtub. Some distributions such as the Weibull distribution, modified Weibull distribution, and extended Weibull distribution with bathtub failure rate function have been used.

Mark Bebbington et al. in 2008 explained that the turning point of the hazard rate function is useful in assessing risks during the productive life stage, and helps in determining and planning appropriate combustion, maintenance and repair policies and strategies. In many bathtub-shaped distributions, the turning point is unique, the risk is slightly different in the productive life stage. The performance of an empirical turning point estimator was studied for the modified Weibull distribution, a bathtub-shaped generalization of the Weibull distribution, and found to be useful in reliability engineering and other fields related to productive life data. An example was applied, and a simulation study was conducted to evaluate the estimator's performance in practice.

In 2013, R.jiang presented two parametric methods to determine the change points of the bathtub curve. The proposed methods are based on the limited-support bathtub curve model. These methods were illustrated through examples with real data. In the same year, Jiang addressed the main characteristics of the bathtub-shaped rate and determined the change points of the curve. She developed non-parametric and parametric methods to estimate the change points and derived simple relationships to estimate the failure rate between the stages of infant mortality and random failure. The proposed methods are useful for model selection and can be used to evaluate the suitability of any model for a bathtub curve, using real-world data as an example to illustrate the proposed methods.

In 2017, Robab Aghazdeh, Mohammad Pourgol, and Karnran Sepanloo studied the time interval of the hazard function in the form of a bathtub, where the two change points were calculated for the stages of combustion and productive life, in addition to the stages of productive life and corrosion. Two criteria were used in this study to determine the change point.

1- Minimum of the hazard function

2- Maximum change in the slope of the hazard function

Parametric methods were used to estimate the parameter, including the Bayes method. The modified Weibull distribution was a suitable model for simulating the three life stages of the bathtub curve.

5- How to test the shape of the bathtub curve for a hazard rate function

Magne Vollen Aarset (1987) explained that the concept of Total Time on Test (TTT) is a useful tool in many reliability cases. In his research, he presented a new test statistic based on the TTT scheme to test whether the generated random sample has a constant life distribution versus a bathtub-shaped hazard rate.

In 1996, M. Xie & C.D.Lai used the method of plotting the hazard rate function over time for the model consisting of adding two survival functions to the Weibull distribution to prove that the hazard rate function of the proposed model has a bathtub shape. (33)

M.Xie, T.N.Goh & P.Ranjan (2002) presented a study that discusses some mathematical models for failure rate functions and proves that they have a bathtub shape using the first derivative.

Alvereze-Alvarado & D.Jayaweera M.S. (2018) presented a mathematical model based on the Markov process to represent the failure rate function over time and prove that it has the shape of a bathtub by using this method and analyzing the reliability of repairable components.

6- The effect of bathtub curve on preventive maintenance

Preventive maintenance is an important and necessary method for ensuring that systems continue to operate efficiently and reduce unexpected breakdowns. The bathtub curve is one of the basic analytical tools used to understand breakdown behavior and plan preventive maintenance work. There are studies that have shed light on the relationship between bathtub characteristics and maintenance planning. These studies include:

Yann Dijoux (2009) presented a new reliability model for complex repairable systems, combining bathtub aging and imperfect maintenance. The bathtub's initial density function allows for the combustion period, productive life, and system depreciation to be taken into accoun. The impact of repair is expressed through a reduction in the system's lifetime, which depends on the system's aging. The main properties of the model are extracted, the most important of which is that efficient maintenance extends the system's useful life. A statistical analysis of the model and its application to actual failure data are also presented.

Makram Krit & Abdelwahab Rebai (2011) conducted an experimental study through which they proposed a framework to determine the characteristics of the failure process and its impact on the maintenance process by formulating the Brown-Proschan model using the bathtub failure severity and the maximum likelihood method was used to estimate the maintenance efficiency.

In 2013, the researchers proposed a general model to illustrate the combined effect of corrective and preventive maintenance on repairable systems. The severity of failures without maintenance is described as a bathtub. The extent of the maintenance effect is expressed by the change in failure severity before and after maintenance. It takes into account the possibility of independent maintenance times with different effects. Probability functions are derived, allowing for parameter estimation and maintenance efficiency evaluation. The properties of parameter estimators are studied theoretically. Finally, the results are applied to a real maintenance dataset.

Harish Kumar N.S., R.P.Choudhry and Ch.S.N. Murthy (2018)showed that the bathtub-shaped failure rate function model is useful in analyzing the reliability of any system, especially in reliability-related preventive maintenance. However, the usual Weibull distribution cannot model the entire life cycle of a system with a bathtub-shaped failure rate function. In this research, the failure rate and reliability of a Komatsu hydraulic bulldozer in surface mines were analyzed, with the aim of improving reliability and reducing the failure rate of each bulldozer subsystem based on preventive maintenance. The bathtub-shaped bucket model can also be considered a simplification of the Weibull distribution.

In the same year, Xuejiao Du et al developed a two-sector fault severity model based on a segmental heterogeneous Poisson process. This model is capable of analyzing repairable systems with bathtub-shaped fault severity. This model takes into account minimal maintenance activities and maintains the chronology of failures over the entire life cycle. The advantages of this model lie in its flexibility in describing the severity of monotonous and non-monotonic failures, and its practicality in determining the start-up or replacement time for repairable systems. Three real-life failure datasets were applied to illustrate the developed model. The results show that the model performs well with respect to the Akaike Information Criterion, mean squared errors, and Cramer-von Mises values. .

7.The Bathtub Curve within Fuzzy Reliability Framework

With the complexity of engineering systems and uncertainty in data and measurements, it has become necessary to develop more flexible models. From here, the idea of the fuzzy bathtub curve emerged, which combines the traditional bathtub model with fuzzy logic techniques to represent ambiguity and imprecision in data, providing a more realistic analysis and adapting to real-world situations that are difficult to model with traditional mathematical accuracy.

M.Shafiq and R.Viertl confirmed in 2017 that with the emergence of recent advances in measurement, not all continuous measurements can be measured as precise numbers, but rather are somewhat fuzzy. Product life is also a continuous phenomenon, and it has already been proven that observations of product or device life are not precise measurements, but rather fuzzy. Therefore, the corresponding analysis techniques used on the data require taking the fuzziness of the observations into account to obtain appropriate estimates. In this study, generalized parameter and hazard rate estimators for bathtub failure rate distributions are proposed to effectively model fuzzy service life data.

Mohammad Nadiafi and Mohamad Ali Farsi (2018) presented a new method for determining failure time distribution functions using probability theory. For this purpose, fuzzy bathtub distributions are generated using expert opinions for key events, and fuzzy formulas are derived for static and dynamic gate failure tree constructions. This process is performed through the proposed fuzzy Monte Carlo simulation over the preferred operating time, using real-time data until failure occurs. Based on this, a major event failure curve and system reliability profile are visualized based on the bathtub failure rates of the defuzzified major events. The results show that the proposed method is not only feasible and robust, but also more accurate than probabilistic techniques since the component failure rates follow the true failure distributions. In 2021, the researchers presented papers, the aim of which was to study a specific correlation in the reliability analysis of fault trees with dynamic correlations; it is called the Time Functional Dependency Gateway (TFDEP). In most systems with functional interconnection architectures equipped with a known FDEP gateway, event failures occur in a specific order, and their dependencies are merely normal and sequential, without any time considerations. However, in some specific systems, events are also time-dependent, in addition to the predetermined order, and for each occurrence, the activation/deactivation time of a specific event must be considered. This time requirement requires the development of a new TFDEP gate. Furthermore, for accurate simulation, fuzzy bathtub distributions are used as the failure time parameter, and fuzzy Monte Carlo techniques are used throughout the predetermined runtime in the simulation. Finally, the proposed approach was applied to an aerospace system as an application example, and simulation results showed that the presented technique is a valid approach for analyzing system reliability with timing and events with unknown failure probabilities. (22)

8. Statistical Distributions with Bathtub-Shaped Failure Rate.

Life distributions composed of several components or a combination of more than one distribution usually have a failure rate function that takes the form of a bathtub, which has contributed to the development of distributions that are more flexible and accurate in representing and analyzing failure and reliability data. These distributions include:

In 1980, Hjorth proposed a three-parameter distribution known as (IDB) that allows representing four failure rate cases: increasing, decreasing, constant, and bathtub-shaped.

Magne Vollen Aarset (1987) proposed a distribution that resembles or is similar to the bathtub curve and thus provides the modified extended Weibull distribution (MWE), which is approximately a link between the Weibull distribution and the exponential distribution, as it provides flexibility in the characteristics of the two distributions.

Elvira Haupt & Hendrik Schabe(1992) proposed a new model for the age distribution, BT-Distribution, with a failure rate that takes the form of a bathtub. This model has two parameters that are easy to use for application in reliability engineering. The maximum likelihood method was used to estimate the two model parameters.

Govind S. Mudholka & Deo Kumar Srivastava (1993) presented a study that includes a proposal for a distribution, which is a simple generalization of the Weibull distribution. This distribution is quite suitable for modeling the age data of bathtub failure rate and for testing the suitability of the Weibull distribution and negative exponential models as sub-hypotheses.

M. Xie & C.D. Lai (1995) proposed in their research a simple model based on combining two failure rate functions of the Weibull distribution to obtain a new distribution called the ensemble Weibull distribution (AWD). The two researches used the graphical method to estimate the parameters of the proposed distribution.

F.K.Wang (2000) proposed a simple distribution by adding two Burr XII distributions to obtain a distribution in which the failure rate function takes a basin-shaped form. It was proposed to model the failure rate function for many mechanical and electronic components and to use graphical estimation and Akaike Information criterion to judge the adequacy of the presented models. This study relied on the graphical method to illustrate the various forms of the failure rate.

C.D.Lai and D.N.P.Murthy (2003) proposed a new life expectancy distribution, the New Modified Weibull Distribution (NMWD). This distribution is capable of modeling a bathtub-shaped hazard rate function. The proposed model is derived as an extreme case of the integrated beta model, and includes both the Weibull distribution and the Type I extreme value distribution as special cases. This model can be considered another useful three-parameter generalization of the Weibull distribution. An advantage of this model is that its parameters can be easily estimated based on the Weibull probability (WPP) plot, which is used as a model identification tool, the percentile method, and the maximum likelihood method. The model's characterization is studied based on the Weibull probability plot. A numerical example is presented and compared with another Weibull extension, the exponential Weibull. The proposed model is compared with other competing models for fitting data that exhibit a bathtub-shaped hazard rate function. (4)

Ammar M.Sarhan & Mazen Zaindin (2009) presented a study in which they proposed a new distribution called the modified Weibull distribution (SZMW). The properties of this distribution were discussed and its unknown parameters were estimated using the maximum likelihood method. The analysis was done on real data .

A new distribution, the new Weibull-Pareto distribution, was defined and studied by Suleman Nasiru and Albert Luguterah in 2015. Various properties of the distributions were obtained, and the maximum likelihood method was used to estimate the distribution's parameters. The usefulness of the distribution was demonstrated by applying it to real data, and the hazard function was plotted and found to take the shape of a bathtub.

Xiaohong Wang, ChuangYu, and Yuxiang Li (2015) collaborated to develop a new four-parameter model to describe FIRE (finite-period distribution model used in reliability engineering) bathtub tools, which is widely used in the concept of component life analysis. The model's parameters were then explained, and a method for estimating the parameters was presented, along with practical examples, using some known operational life data. By comparing the new model with some existing bathtub curve models, we found the new model easy to use and clear in its parameters. Moreover, this model has universal applicability, not only fitting bathtub-shaped failure rate curves, but also applying to constant, increasing, and decreasing failure rate curves.

M.H.Tahir et al.(2016) proposed a three-parameter Weibull-Pareto distribution, which can produce the most important forms of hazard rate, namely constant, increasing, decreasing, bathtub, and inverted bathtub. Various structural properties of the new distribution are derived, including explicit expressions for moments and incomplete moments, Bonferroni and Lorenz curves, mean deviations, mean residual life, mean waiting time, generating functions, and quantum functions. The model parameters were estimated using the maximum likelihood method. The usefulness of the new model was demonstrated on two real-world data sets on the Whitton River flood and bladder cancer. In both applications, the new model provides a better fit than the Komaraswamy-Pareto, Beta-Pareto, Exponential Pareto, and Pareto models.

In 2017, Ammar M. Sarhan proposed a two-parameter discrete distribution, based on the two-parameter continuous bathtub distribution (TPBT). It is the only two-parameter discrete distribution that exhibits a bathtub-shaped hazard function.

The paper discusses some statistical properties of the distribution. Three different methods are used to estimate the two unknown parameters. The point estimators for the coefficients are not closed-form. The resampling method is used to estimate the distributions of these estimators. Various approximations for the interval estimates for the two parameters are discussed. Real data sets are analyzed to demonstrate how this distribution works in practice. A simulation study is conducted to verify the properties of the obtained estimates and compare their performance. In 2019, the researcher estimated the parameters of this distribution and reliability measures using the Bayesian method, as he was unable to obtain a suitable joint posterior distribution of the model parameters. Therefore, numerical techniques were needed. Four Bayesian numerical methods were applied to obtain random results from the joint posterior distribution for use in estimating model parameters and reliability measures without deriving the actual joint posterior distribution. Here, the two model parameters were assumed to be independent random variables with beta and gamma distributions. Two real data sets were analyzed using the Bayesian techniques applied here. A simulation study was conducted to verify the properties of the applied methods. In the same year, Jamal.N.Al-Abbasi and others proposed a three-parameter lifetime distribution, called the generalized Weibull uniform distribution, which is an extension of the Weibull distribution. This distribution has a bathtub-shaped failure rate function, which enables it to fit real lifetime data sets. Various structural properties of the new distribution are derived, including the quantile function, moments, moment-generating function, and rank statistics. The parameters of the proposed distribution are estimated using the maximum likelihood method, and the performance of the maximum likelihood estimation is evaluated using simulations. Applications to real-world data show that the proposed distribution can be very useful in fitting real data.

L.C. Mendez et al. (2022) proposed the ensemble Perks distribution (APD), where they reviewed the properties of this distribution and the maximum likelihood method was used to estimate the distribution parameters with the estimation of the reliability function. Several criteria were used for comparison with a group of distributions such as the ensemble Weibull distribution (AWD), the generalized Perks distribution (EPD), the modified Weibull distribution (MWE), and other distributions that describe the failure rate in the form of a bathtub.

In 2023, a new distribution was proposed by Luis Carlos et al. who combined the Chen distribution and the Perks distribution to obtain the Chen-perks distribution (CPD). The parameters of the proposed distribution and the reliability function were estimated using the maximum likelihood method, and statistical analysis was performed in three case studies to determine the behavior of this distribution compared to other distributions that have a failure rate function that takes the form of a bathtub. In the same year, researcher Laila A. Al-Essa and others presented a new flexible distribution with four parameters, by converging between the exponential distribution and the Weibull distribution using the single function transformation, which provides greater flexibility in terms of fitting, and is called the modified exponential Weibull distribution (MEW). The MEW model was designed to provide a more accurate description of the failure time data generated by a system with one or more failure modes. Due to its complexity, it has a hazard rate similar to a bathtub. The properties of moments, the quantum function, and the remaining life are derived and discussed. The failure rate function and several distributional properties of the MEW model are also discussed. The maximum likelihood method and the Bayesian method are used to estimate unknown parameters.The Hamiltonian Monte Carlo (HMC) algorithm was used to simulate posterior distributions and verify the Bayesian estimators for MEW. We examined the behavior of the MEW model on two bathtub-shaped hazard rate datasets and compared it with five other popular bathtub-shaped methodologies. The results indicate that the MEW model provided the best description of the two failure time datasets. This indicates that the proposed model may be an effective candidate for solving many real-life problems.

Mustapha Muhammad et al.(2024) presented in this paper a new three-parameter model, with an increasing function and a bathtub failure rate function, as an extended model of the Mustafa distribution of the second type (Mu-II). This model can be very useful in statistical studies, reliability, computer science, and engineering. Various mathematical and statistical properties of distributions are discussed, such as moments, mean deviations, Bonferroni and Lorenz curves, entropy, rank statistics, and extreme value distributions. Statistical inferences are discussed using the maximum likelihood method and evaluated through simulation studies. The new model provides a better fit than some other existing distributions, as measured by some model selection criteria and goodness-of-fit statistics.

9. Theoretical and applied criticisms of the bathtub curve.

The typical bathtub curve and its "standard" shape have gained widespread acceptance as an engineering tool in reliability analysis. However, some recent research reveals that the typical theory of the "bathtub curve" is not convincing in some cases. Therefore, a critical discussion has been developed to examine the bathtub curve over three important bathtub periods: infant mortality, useful life, and wearout.

Ronald W. Smith and Duane L. Dietrich (1994) presented a practical interpretation of the bathtub curve, where the failure rate is governed by a physical process of deterioration of the strength of components over time as a result of stress. Furthermore, this physical interpretation of the hazard rate rejects the traditional, purely statistical view that the bathtub curve can be divided into early, steady, and corrosion failures, and instead supports the more recent claim that most failures are Caused by corrosion.

George-Ann Klutke et al. (2003) addressed some of the basic assumptions underlying the bathtub curve. It has been shown that any practical hazard function is unlikely to decrease to near zero. Therefore, extreme caution should be exercised when interpreting the hazard function, particularly when applying quality control practices, such as internal combustion or environmental stress shielding, to manufactured products.He added that using the h(t) curve only when studying the characteristics of the early stages of a system's life cycle, consisting of a mixture of subsets, can be highly misleading. In particular, the bathtub curve should not be used to describe the behavior of such groups in the early stages of their life cycle .Instead, practitioners should pay special attention to the probability density function (pdf) of the age of the system population whenever it is believed that a mixture may include subsets of so-called “weak” or “exotic” devices that lead to early failures in the total system population.

Tan Cheng (2006) discussed in his research the bathtub theory and how to define the three stages that make up this curve geometrically. He added that it is necessary to integrate the methods used to identify and eliminate faults together, and apply them as a coherent program for quality and reliability aimed at actually achieving the required function, the conditions stated, and the specified time period, taking into account costs, customer requirements, design, work and manufacturing practices.

10. Conclusions

The bathtub curve is one of the important tools and methods in reliability analysis to describe the behavior of failure rates over time for products and systems. It appears in three distinct stages: the early failure stage, the productive life stage, and the workout stage, despite its theoretical and practical importance, recent studies have shown that this curve is not comprehensive for all systems and does not reflect the complex systems of failure rates in modern systems, which calls for the development of more flexible models.

1-The studies reviewed indicated that intelligent methods should be integrated to improve the accuracy of representation and prediction, particularly the introduction of artificial intelligence algorithms into bathtub curve models. Combining artificial intelligence and curve models leads to hybrid predictive models and provides accurate and unconventional solutions.

2-It can be said that the bathtub curve has been the subject of much research, but very little research has used fuzzy logic, which provides a flexible framework for dealing with data inaccuracy or ambiguity in human assessment. Fuzzy functions should be used to estimate the failure rate at each stage, rather than vague values.

3-It is worth noting that most of the studies reviewed use the same data for comparison purposes as real data and do not adopt recent data through which the behavior of the curve can be studied.

4-The bathtub curve is considered one of the important methods in its ability to support preventive maintenance decisions and improves system design. Therefore, it was important to adopt nonparametric models to estimate the failure rate in fault contexts. It provides a flexible and reliable description of the shape of the bathtub curve without prior knowledge about the data distribution using nonparametric tools, such as Kaplan-Meier and Nelson-Aalen, with kernel hazard estimation, to identify the three phases of this curve and detect change points with greater accuracy than parametric methods. Therefore, it is preferable to use hybrid models that combine nonparametric estimation for shape detection with a semiparametric or parametric model for prediction and decision-making.

5- The bathtub curve remains an important tool for understanding failure behavior across the system lifecycle. Combining this curve with modern statistical methods and artificial intelligence enhances the accuracy and flexibility of modeling, particularly when dealing with realistic or incomplete field data. The importance of this combination lies in its ability to reveal the curve's characteristics and identify change points without prior assumptions.

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