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Topp-Leone Exponentiated Burr XII Distribution: Theory and Application to Real-Life Data Sets

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ABSTRACT

This paper introduces a novel compound distribution, named the Topp Leone Exponentiated Burr XII (TLEBXII) Distribution, aimed at generating a more flexible family of probability distributions. By taking the Exponentiated Burr XII distribution as the base and the Topp Leone exponentiated G family as the generator, this new distribution seeks to enhance modeling capabilities in various statistical applications. We present general expressions for the density and distribution functions of the TLEBXII model and explore its fundamental properties, including survival and hazard rate functions, moments, moment-generating functions, and entropy. Additionally, we discuss the methodology for estimating the model parameters through maximum likelihood estimation. The utility and superior performance of the TLEBXII distribution are demonstrated through its application to two real datasets, where it shows significant improvements over existing models. This paper underscores the potential of the TLEBXII distribution in providing a robust tool for statistical modeling and analysis.

1. Introduction

Classical statistical distributions are fundamental to data analysis but often struggle to model datasets with characteristics that significantly deviate from normality. These limitations are evident in their failure to capture the complexities of datasets characterized by pronounced skewness, heavy tails, or multimodality, among other non-normal features. To address these challenges, researchers have increasingly adopted the strategy of compounding classical distributions with a diverse family of distributions. This approach not only imbues classical models with enhanced flexibility but also significantly improves their ability to accurately represent the intricate patterns of non-normally distributed data. Despite the progress achieved with these compounded models, as highlighted in recent literature [1] – [9], finding a universal solution has proven elusive. The vast variability in data complexity across various contexts highlights the need for a broad spectrum of compound distributions. This diversity ensures that data analysts have a comprehensive toolkit from which they can select the most appropriate model to represent their unique datasets accurately. A popular method for compounding distributions involves integrating a baseline distribution with a generator, thereby inducing additional parameters into the baseline distribution. This process, which increases the distribution's skewness, is known as extending, enhancing, or generalizing a baseline or classical distribution. Notable families of distributions introduced in the literature include [10] – [14]. This study focuses on the two-shape parameter Burr XII distribution, which has been extensively extended by researchers [15] –

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[20]. We consider the Topp-Leone Exponentiated Family of distributions, introduced by [21], which also features two shape parameters. By compounding the Burr XII distribution with the Topp-Leone Exponentiated generator, we aim to develop a new hybrid distribution with four shape parameters. This distribution is expected to possess the necessary flexibility to fit datasets with non-normal features, such as skewness. Therefore, the aim of this study is to contribute to the evolving landscape of statistical models by developing a novel compound distribution. Through this endeavour, we seek to expand the arsenal of tools available for addressing the challenges posed by non-normal data features, thereby offering more tailored and precise analytical capabilities.

2. Formulation of the New Model

The CDF and PDF of the Topp Leone Exponentiated generator developed by [21] are expressed as:

$$F(x,\gamma,\theta) = (1-(1-G(x)^{\gamma})^2)^{\theta} \qquad (1)$$

$$f(x,\gamma,\theta) = 2\gamma\theta g(x)G(x)^{\gamma-1}(1-G(x)^{\gamma})(1-(1-G(x)^{\gamma})^2)^{\theta-1} \qquad (2)$$

where $\gamma > 0$ and $\theta > 0$ denote the shape parameters.

The CDF and the PDF of the Burr XII distribution is given as:

$$G(x) = 1 - \frac{1}{(1+x^a)^b} \qquad (3)$$

$$g(x) = \frac{abx^{b-1}}{(1+x^a)^{b+1}}, \qquad x > 0 \qquad (4)$$

where $a > 0$ and $b > 0$ denote the shape parameters.

By substituting (3) into (1), we have the CDF of the TLEBXII distribution given as:

$$F(x) = \left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta} \qquad (5)$$

And by substituting (3) and (4) in (2), the corresponding PDF is given as

$$f(x) = \frac{2\gamma\theta abx^{a-1}}{(1+x^a)^{b+1}}\left[1 - \frac{1}{(1+x^a)^b}\right]^{\gamma-1}\left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma}\right]\left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta-1} \qquad (6)$$

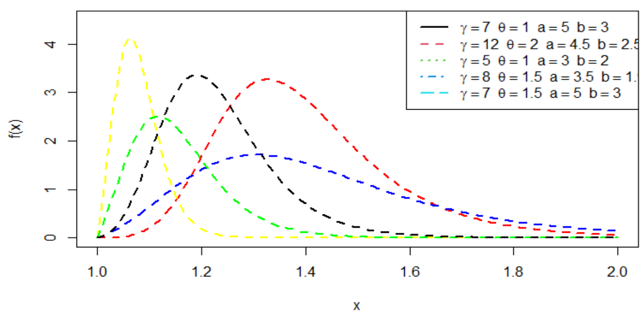

Figure 1. PDF plot of the TLEBXII distribution

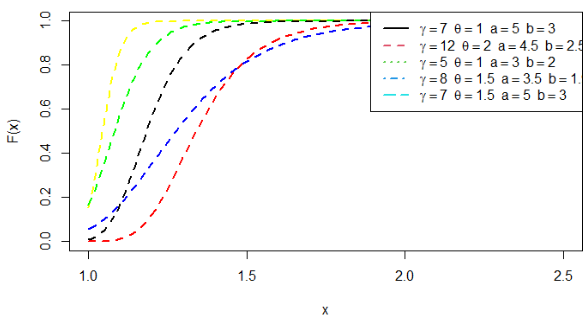

Figure 2. CDF plot of the TLEBXII distribution

3. Mathematical Properties

3.1 Survival function

By definition, the survival function is mathematically given as

$$S(x) = 1 - F(x)$$

The survival function of the TLEBXII distribution is given by:

$$S(x) = 1 - \left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^\gamma\right]^2\right\}^\theta \tag{7}$$

3.2 Hazard Function

The hazard function h(x) is generally defined mathematically as:

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)}$$

For the TLEBXII Distribution, the h(x) is given by

$$h(x) = \frac{\frac{2\gamma\theta abx^{a-1}}{(1+x^a)^{b+1}}\left[1 - \frac{1}{(1+x^a)^b}\right]^{\gamma-1}\left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^\gamma\right]\left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^\gamma\right]^2\right\}^{\theta-1}}{1 - \left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^\gamma\right]^2\right\}^\theta} \tag{8}$$

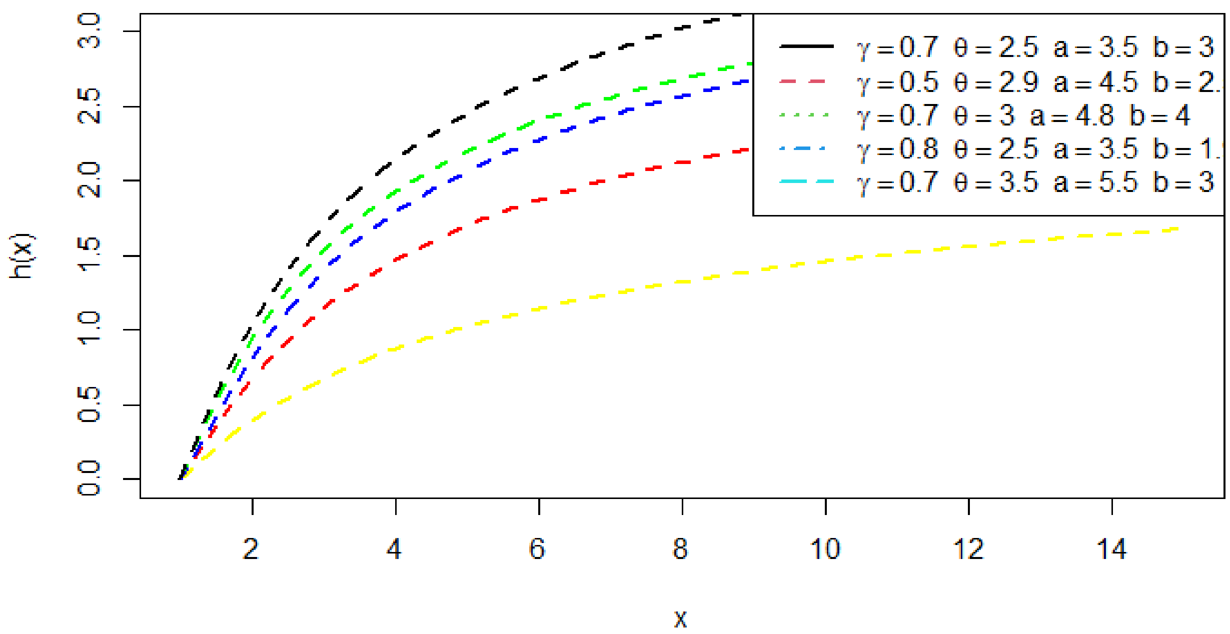

Figure 3. Hazard Plot of the TLEBXII distribution

3.3 Reverse Hazard Function

The reverse hazard function r(x) is defined as:

$$r(x) = \frac{f(x)}{F(x)}$$

The r(x) of the TLEBXII distribution is expressed as:

$$r(x) = \frac{\frac{2\gamma\theta abx^{a-1}}{(1+x^a)^{b+1}}\left[1 - \frac{1}{(1+x^a)^b}\right]^{\gamma-1}\left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^\gamma\right]\left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^\gamma\right]^2\right\}^{\theta-1}}{\left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^\gamma\right]^2\right\}^\theta}$$

$$r(x)=\frac{\frac{2\gamma\theta abx^{a-1}}{(1+x^a)^{b+1}}\left[1-\frac{1}{(1+x^a)^b}\right]^{\gamma-1}\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]}{\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}} \tag{9}$$

3.4 The Cumulative Hazard Function

The cumulative hazard function is given by:

$$H(x)=\int_0^x \frac{f(x)}{F(x)}\,dx$$

The cumulative hazard function of the Topp-Leone Exponentiated Burr XII is given by:

$$H(x)=\int_0^x \frac{\frac{2\gamma\theta abx^{a-1}}{(1+x^a)^{b+1}}\left[1-\frac{1}{(1+x^a)^b}\right]^{\gamma-1}\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta-1}}{1-\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta}}\,dx$$

$$H(x)=-\ln\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta} \tag{10}$$

3.5 Quantile function

This function is obtained by inverting the CDF of any given continuous probability distribution. The q^{th} quantile u_q of the TLEBXII distribution can be obtained as follows:

$$u_q=\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta}$$

Which gives

$$x_q=\left\{\left[1-\left(1-\left[1-u^{1/\theta}\right]^{1/2}\right)^{1/\gamma}\right]^{-1/b}-1\right\}^{1/a} \tag{11}$$

3.6 Expansion of the PDF and CDF

The PDF of the TLEBXII model can be reduced using the power series expansion as follows:

$$f(x)=\frac{2\gamma\theta abx^{a-1}}{(1+x^a)^{b+1}}\left[1-\frac{1}{(1+x^a)^b}\right]^{\gamma-1}\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta-1}$$

$$\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\theta-1}{i}\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^{2i}$$

$$\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^{2i}=(-1)^j\binom{2i+1}{j}\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma j}$$

$$\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma j+\gamma-1}=\sum_{k=0}^{\infty}(-1)^k\binom{\gamma j+\gamma-1}{k}(1+x^a)^{-bk}$$

$$(1+x^a)^{-(bk+b-1)}=\sum_{l=0}^{\infty}(-1)^l\binom{bk+b-l}{l}x^{al+a-1}$$

$$f(x)=\sum_{i=0}^{\infty}2\gamma\theta ab\,(-1)^{i+j+k+l}\binom{\theta-1}{i}\binom{2i+1}{j}\binom{\gamma j+\gamma-1}{k}\binom{bk+b-l}{l}x^{al+a-1}$$

Therefore,

$$f(x) = \sum_{i=0}^{\infty} \Upsilon_{i,j,k,l}\, x^{al+a-1} \tag{12}$$

Where

$$\Upsilon_{i,j,k,l} = 2\gamma\theta ab\,(-1)^{i+j+k+l} \binom{\theta-1}{i}\binom{2i+1}{j}\binom{\gamma j+\gamma-1}{k}\binom{bk+b-l}{l}$$

The expansion of the CDF:

$$F(x) = \left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta}$$

$$\left\{1 - \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\}^{\theta} = \sum_{m=0}^{\infty} (-1)^m \binom{\theta}{m} \left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^{2m}$$

$$\left[1 - \left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^{2m} = \sum_{u=0}^{\infty} (-1)^n \binom{2m}{n} \left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma n}$$

$$\left(1 - \frac{1}{(1+x^a)^b}\right)^{\gamma n} = \sum_{p=0}^{\infty} (-1)^p \binom{\gamma n}{p} (1+x^a)^{-bp}$$

$$(1+x^a)^{-bp} = \sum_{q=0}^{\infty} (-1)^q \binom{bp-q-1}{q} x^{aq}$$

The CDF can also be expressed as:

$$F(x) = \sum_{m,n,p,q=0}^{\infty} (-1)^{m+n+p+q} \binom{\theta}{m}\binom{2m}{n}\binom{\gamma n}{p}\binom{bp-q+1}{q} x^{aq}$$

Therefore, $\mathcal{B}\, x^{aq}$

$$F(x) = \sum_{m,n,p,q=0}^{\infty} \mathcal{B}\, x^{aq} \tag{13}$$

Where

$$\mathcal{B} = (-1)^{m+n+p+q} \binom{\theta}{m}\binom{2m}{n}\binom{\gamma n}{p}\binom{bp-q+1}{q}$$

3.8 Moments

Moment analysis is a crucial aspect of statistical analysis, particularly in its applications across various fields. Moments offer a systematic way to capture and describe the essential features of a probability distribution. Specifically, they are used to quantify the central tendency, dispersion, skewness, and kurtosis of distributions. These characteristics provide insights into the shape and behaviour of the distribution, which are invaluable for statistical modelling, inference, and decision-making processes. For the TLEBXII distribution, determining the rth moment is essential for understanding its unique properties and behaviour. The rth moment is defined as

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx = \sum_{i=0}^{\infty} \Upsilon_{i,j,k,l} \int_0^{\infty} x^{r+al+a-1} dx$$

$$\mu_r' = \sum_{i=0}^{\infty} \Upsilon_{i,j,k,l}\, \psi \tag{14}$$

Where

$$\psi = \int_0^\infty x^{r+al+a-1}dx$$

3.9 Moment Generating Function

$$E(e^{tx}) = \sum_{i=0}^{\infty} \Upsilon_{i,j,k,l} \int_{-\infty}^{\infty} e^{tx}x^{r+al+a-1}dx$$

$$E(e^{tx}) = E(e^{tx}) = \sum_{i=0}^{\infty} \Upsilon_{i,j,k,l}\,\phi \tag{15}$$

Where

$$\phi = \int_{-\infty}^{\infty} e^{tx}x^{r+al+a-1}dx$$

3.9 The Entropy

Entropy serves as a measure of uncertainty present in a random observation from its actual population. A larger entropy value indicates greater uncertainty within the data. For continuous probability distributions, the concept of differential entropy is often considered the most effective measure. This can be seen as an extension of Shannon's entropy, defined for a true continuous random variable X as:

$$I_y(x) = \frac{1}{1-\lambda}\log\int_{-\infty}^{\infty} f(x)^{\lambda}dx$$

The entropy of the TLEB distribution is given by:

$$f(x)^{\lambda} = \left(\sum_{i=0}^{\infty} \Upsilon_{i,j,k,l}\, x^{al+a-1}\right)^{\lambda}$$

$$f(x)^{\lambda} = \left(\sum_{i=0}^{\infty} \Upsilon_{i,j,k,l}\right)^{\lambda} (x^{al+a-1})^{\lambda}$$

$$f(x)^{\lambda} = \left(\sum_{i=0}^{\infty} \Upsilon_{i,j,k,l}\right)^{\lambda} (\Psi)^{\lambda}$$

$$\Psi = x^{al+a-1}$$

Therefore, the entropy is given by:

$$I_y(x) = \frac{1}{1-\lambda}\left[\left(\sum_{i=0}^{\infty} \Upsilon_{i,j,k,l}\right)^{\lambda} \log\int_{-\infty}^{\infty}\Psi^{\lambda}dx\right] \tag{16}$$

4. Parameter Estimation

The parameters of the TLEBXII distribution are obtained using the maximum likelihood technique. The likelihood function of the Topp-Leone Exponentiated Burr XII Distribution is given by:

$$\tau = n\log 2 + n\log\theta + n\log\gamma + n\log b + n\log a + (\gamma-1)\sum_{i=1}^{n}\log\left(1-\frac{1}{(1+x^a)^b}\right)$$
$$+\sum_{i=1}^{n}\log\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right] + (\theta-1)\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\} \tag{17}$$

Differentiating equation (17) with respect to a give:

$$\frac{\partial\tau}{\partial a}=\frac{n}{a}+(\gamma-1)\sum_{i=1}^{n}\left(\frac{abx^{a-1}(1+x^a)^{-(b+1)}}{1-(1+x^a)^{-b}}\right)-\sum_{i=1}^{n}\left(\frac{ab\gamma(1+x^a)^{-(b+1)}(1-(1+x^a)^{-b})}{(1-(1+x^a)^{-b})^{-(\gamma-1)}}\right)$$
$$+(\theta-1)\sum_{i=1}^{n}\left(\frac{2ab\gamma x^{a-1}(1+x^a)^{-(b+1)}(1-(1+x^a)^{-b})^{\gamma-1}\left\{[1-(1+x^a)^{-b}]^{\gamma}\right\}}{\left\{1-\left[1-(1-(1+x^a)^{-b})^{\gamma}\right]^2\right\}}\right) \quad (18)$$

Differentiation with respect to b gives:

$$\frac{\partial\tau}{\partial b}=\frac{n}{b}-(\gamma-1)\sum_{i=1}^{n}\left(\frac{(1+x^a)^{-b}\log(1+x^a)}{1-(1+x^a)^{-b}}\right)-\sum_{i=1}^{n}\left(\frac{\gamma(1-(1+x^a)^{-b})^{\gamma-1}(1+x^a)^{-b}\log(1+x^a)}{(1-(1+x^a)^{-b})^{\gamma-1}}\right)$$
$$+(\theta-1)\sum_{i=1}^{n}\left(\frac{2\gamma(1+x^a)^{-b}\left[1-(1-(1+x^a)^{-b})^{\gamma}\right](1-(1+x^a)^{-b})^{\gamma-1}\log(1+x^a)}{1-\left[1-(1-(1+x^a)^{-b})^{\gamma}\right]^2}\right) \quad (19)$$

Differentiating with respect to γ gives:

$$\frac{\partial\tau}{\partial\gamma}=\frac{n}{\gamma}+\sum_{i=1}^{n}\log\left(1-(1+x^a)^{-b}\right)-\sum_{i=1}^{n}\left(\frac{(1-(1+x^a)^{-b})^{\gamma}\log(1-(1+x^a)^{-b})}{1-\left[1-(1+x^a)^{-b}\right]^{\gamma}}\right)$$
$$+(\theta-1)\sum_{i=1}^{n}\left(\frac{2\left\{1-\left[1-(1+x^a)^{-b}\right]^{\gamma}\right\}\left[1-(1+x^a)^{-b}\right]^{\gamma}\log\left[1-(1+x^a)^{-b}\right]}{1-\left[1-(1-(1+x^a)^{-b})^{\gamma}\right]^2}\right) \quad (20)$$

$$\frac{\partial\tau}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1-\frac{1}{(1+x^a)^b}\right)^{\gamma}\right]^2\right\} \quad (21)$$

Setting equations (18) to (21) equal to zero yields the maximum likelihood estimates for the parameters a, b, γ, and θ, respectively. However, it is challenging to obtain these estimates analytically. Therefore, we used R software to compute the estimates.

5. Application

This section presents applications of the TLEBXII distribution using real data sets. The performance of the newly developed model that is Topp-Leone Exponentiated Burr XII (TLEBXII) is compared with the baseline distribution (Burr XII) as well as other extensions such as Extended Burr XII (ExBXII), Transmuted Burr XII (TBXII) and Topp Leone-Exponential distributions.

First Data Set

The first data set considered in this work consists of 100 observations of breaking stress of carbon fibers given by [22] as presented below:

{0.920, 0.9280, 0.997, 0.9971, 1.0610, 1.117, 1.1620, 1.183, 1.187, 1.1920, 1.196, 1.2130, 1.215, 1.2199, 1.220, 1.2240, 1.225, 1.2280, 1.237, 1.240, 1.244, 1.259, 1.2610, 1.263, 1.276, 1.310, 1.3210, 1.3290, 1.3310, 1.337, 1.351, 1.359, 1.388, 1.4080, 1.449, 1.4497, 1.450, 1.459, 1.471, 1.475, 1.477, 1.480, 1.489, 1.501, 1.507, 1.515, 1.530, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.5620, 1.566, 1.585, 1.586, 1.599, 1.602, 1.6140, 1.6160, 1.617, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.7590, 1.777, 1.7940, 1.799, 1.806, 1.814, 1.8160, 1.8280, 1.830, 1.884, 1.892, 1.944, 1.972, 1.9840, 1.987, 2.02, 2.0304, 2.0290, 2.0350, 2.0370, 2.0430, 2.0460, 2.0590, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.3060}

Second Data Set

The second dataset consists of data on the strength of 1.5 cm glass fibre, comprising 63 observations. This dataset has been previously analyzed by the study referenced as [23], as detailed below:

{0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63,1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82,1.84, 1.84, 2.00, 2.01, 2.24}

We highlight some of the Topp Leone Exponentiated Burr XII distribution's advantages over various related distributions with two and three parameters when it comes to fitting the first and second datasets. The following formulas were used to compare results using the following evaluation metrics.

$AIC = 2(LL) + 2k$

$BIC = -(2 * LL) + (k * (\log(n)))$

$CAIC = -2(LL) + 2k(k+1)/(n-k-1)$

$HQIC = -2(LL) + (2 * k * \log(\log(n)))$

Table 1: Statistics of the two datasets

Data	Minimum	Q_1	Media	Mean	Q_3	Maximum
Dataset I	0.920	1.302	1.544	1.658	1.814	5.306
Dataset II	0.550	1.375	1.590	1.507	1.685	2.240

Table 2: MLE, AIC, CAIC, BIC, and HQIC Values for the two Dataset

Data Set I	a	b	γ	θ	MLE	AIC	$CAIC$	BIC	$HQIC$
TLEBXII	4.0576	0.6722	1.3103	5.2847	52.6045	113.209	113.690	114.479	117.486
ExBXII	19.079	0.1647	−	−	65.0393	347.079	134.202	139.289	136.187
BXII	1.1398	1.8486	−	−	171.529	347.059	347.182	352.269	349.167
T − BXII	1.2587	3.2391	60.440	−	217.000	429.040	428.790	425.829	425.877
TL − Ex	3.1231	0.0966	−	−	150.551	305.103	305.227	310.313	307.212
Data Set 2	a	b	γ	θ	MLE	AIC	$CAIC$	BIC	$HQIC$
TLEBXII	2.5570	1.4897	10.698	0.6009	30.691	69.371	70.069	69.665	72.751
ExBXII	7.1549	0.4892	−	−	40.718	85.436	85.637	89.722	87.124
BXII	1.4885	1.5360	−	−	100.22	204.43	204.63	208.78	208.12
T − BXII	2.0943	1.5360	41.652	−	97.391	188.39	188.38	186.49	186.25
TL − Ex	1.5060	0.2204	−	−	88.830	181.66	181.86	185.95	183.35

Using the two datasets, we fitted the proposed TLEBXII (a, b, γ, θ) distribution, as well as four additional distributions: the Exponentiated Burr XII, the Burr XII, the Transmuted Burr XII, and the Topp-Leone Exponential distribution. To compare the goodness of fit, we present the maximum likelihood estimates for the unknown parameters of these five distributions, along with the values for the AIC, BIC, CAIC, and HQIC. The TLEBXII distribution outperforms its competitors, exhibiting lower values across all the relevant criteria, which indicates its superior fit to the datasets.

6. Conclusions

In this paper, we introduce an innovative extension to the Burr XII distribution, achieved through integration with the Topp-Leone exponentiated G family. This extension significantly enhances the flexibility and applicability of the Burr XII model by introducing additional shape parameters, resulting in a four-parameter distribution. We delve into the mathematical underpinnings of the model, examining its moments, moment-generating function, and entropy characteristics. Parameter estimation for our model employs the Maximum Likelihood Estimation (MLE) technique, chosen for its robustness and efficiency. The practical utility of our extended distribution is underscored through the analysis of two real-world datasets. Comparative performance metrics reveal that our model outperforms other well-established models, showcasing superior fit. This remarkable performance highlights the potential of our extended Burr XII distribution. We anticipate that the versatility and enhanced performance of our model will establish it as a valuable tool in various fields.

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