



A New Expansion of the Inverse Weibull distribution: Properties with Applications

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ABSTRACT

The use of statistical distributions to model life phenomena has received a great deal of attention in various sciences. Recent studies have shown the possibility of statistical distributions in data modeling in applied sciences, especially in environmental sciences. Among them is the inverse Weibull distribution, which is one of the most common statistical models that can be used very effectively in modeling data in the health, engineering, and environmental fields, as well as other fields. This study proposes to present a new generalization for the inverse Weibull distribution, where two new parameters are added to the basic distribution according to the Odd Lomax-G family so that the new generalization is more modern and flexible with real-world data. It is called the Odd Lomax Inverse Weibull (LoIW) distribution. The OLIW distribution comes with an expansion of its pdf and CDF functions by using binomial series, exponential, and Logarithm expansions with many statistical properties such as (Rényi entropy, moments, skewness, kurtosis with the moments generating function (mgf), ordered statistics, as well as the Quantile function), and the four distribution parameters are estimated using the maximum likelihood function (MLEs). To ensure the robustness of the proposed model, a practical application is conducted using the R language on two different types of real data and compared with many other statistical models.

1. Introduction

The great development witnessed by real-world data requires creating models with high-accuracy specifications. Therefore, many researchers in the field of statistics have decided to work on providing new distributions. To give flexibility to the statistical distributions, several parameters were added to the basic distributions to produce quite a few families of statistical distributions that attracted many researchers, for example:

The Gompertz-G (Go-G) family by [1], Marshall- Olkin generalized G Poisson (MOG) family [2], combining Marshal-Olkin transformation (MOW-G) family [3], exponentiated half-logistic-Gompertz Topp-Leone-G (TIEHL-Gom-TL-G) family [4], topp-Leone (NETLE) family [5], and Odd Burr XII Gompertz Distribution (OBXIIGo) family [6]. In this paper, we will introduce a new expansion of the inverse Weibull distribution based on the LoG family, which has the pdf and CDF functions by equation (1) and (2), respectively: [7]

$$F_{Lo-G}(x, a, b, \xi) = 1 - \left(1 - \frac{G(x, \xi) \cdot \log(1 - G(x, \xi))}{b}\right)^{-a} \quad (1)$$

$$f_{Lo-G}(x, a, b, \xi) = \frac{a}{b} g(x, \xi) \left(1 - \frac{G(x, \xi) \cdot \log(1 - G(x, \xi))}{b}\right)^{-(a+1)} \left[\frac{G(x, \xi)}{1 - G(x, \xi)} - \log(1 - G(x, \xi))\right], \quad x, a, b, \xi > 0 \quad (2)$$

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Where $a, b, \xi > 0$ are their parameters, $x > 0$.

2. The OLIW distribution

Take IW the Inverse Weibull dist. as a baseline model, while the pdf and CDF of IW in equations (3) and (4), respectively [8]:

$$G(x, \xi) = e^{-\lambda x^{-\vartheta}} \quad (3)$$

$$g(x, \xi) = \lambda \vartheta x^{-(\vartheta+1)} e^{-\lambda x^{-\vartheta}} \quad (4)$$

$$F_{\text{LoIW}}(x, a, b, \lambda, \vartheta) = 1 - \left(1 - \frac{e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})}{b} \right)^{-a} \quad (5)$$

While when substitute equation (4) into equation (2), we get pdf of the Odd Lomax

When substitute equation (3) into equation (1), we get CDF of the Odd Lomax Inverse Weibull (LoIW) distribution of the form:

Inverse Weibull (LoIW) distribution of the form:

$$f_{\text{LoIW}}(x, a, b, \lambda, \vartheta) = \frac{a\lambda\vartheta}{b} x^{-(\vartheta+1)} e^{-\lambda x^{-\vartheta}} \left(1 - \frac{e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})}{b} \right)^{-(a+1)} \left[\frac{e^{-\lambda x^{-\vartheta}}}{1 - e^{-\lambda x^{-\vartheta}}} - \log(1 - e^{-\lambda x^{-\vartheta}}) \right], \quad x, a, b, \lambda, \vartheta > 0 \quad (6)$$

The Survival function can be obtained from the following equation [9]:

$$S(x)_{\text{LoIW}} = 1 - F_{\text{OLE}}(x, a, b, \lambda, \vartheta)$$

Substituting equation (5) we get:

$$S(x)_{\text{LoIW}} = \left(1 - \frac{e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})}{b} \right)^{-a} \quad (7)$$

While the hazard functions of the LoIW dist. is obtained by Equation:

$$h(x)_{\text{LoIW}} = \frac{pdf}{S(x)} = \frac{f_{\text{LoIW}}(x, a, b, \lambda, \vartheta)}{S(x)_{\text{LoIW}}} = \frac{a\lambda\vartheta x^{-(\vartheta+1)} e^{-\lambda x^{-\vartheta}} \left[\frac{e^{-\lambda x^{-\vartheta}}}{1 - e^{-\lambda x^{-\vartheta}}} - \log(1 - e^{-\lambda x^{-\vartheta}}) \right]}{b - e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})} \quad (8)$$

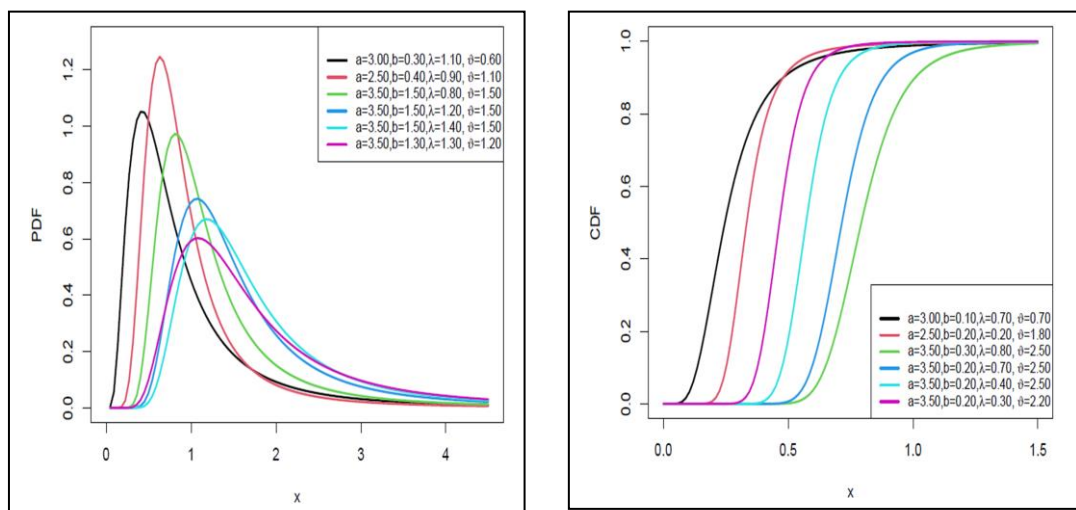


Figure 1. plot PDF and CDF for the LoIW distribution.

3. Mathematical Properties of LoIW

3.1 Useful representations pdf and CDF

We can expand the CDF of LoIW distribution using equation (5) as follows: Using binomial series expansion we get:

$$\left(1 - \frac{e^{-\lambda x^{-\vartheta}} \log(1 - e^{-\lambda x^{-\vartheta}})}{b}\right)^{-a} = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)b^{-k}}{k! \Gamma(a)} e^{-k\lambda x^{-\vartheta}} (\log(1 - e^{-\lambda x^{-\vartheta}}))^k$$

Also by using logarithm expansion of $[\log(1 - e^{-\lambda x^{-\vartheta}})]^k$ by the form:

$$[\log(1 - e^{-\lambda x^{-\vartheta}})]^k = \sum_{i=0}^{\infty} (-1)^i d_{k,i} (e^{-\lambda x^{-\vartheta}})^{i+k} = \sum_{i=0}^{\infty} (-1)^i d_{k,i} e^{-(i+k)\lambda x^{-\vartheta}}$$

Where $d_{k,i} = \frac{1}{i} \sum_{m=1}^i \frac{[m(k+1)-i]}{m+1}$ for $i \geq 0$ and $d_{k,0} = 1$

$$F_{\text{LoIW}}(x, a, b, \lambda, \vartheta) = 1 - \sum_{k=i=0}^{\infty} \frac{\Gamma(a+k)b^{-k}(-1)^i d_{k,i}}{K! \Gamma(a)} e^{-(i+2k)\lambda x^{-\vartheta}}$$

Take

$$\Upsilon = \sum_{k=i=0}^{\infty} \frac{\Gamma(a+k)b^{-k}(-1)^i d_{k,i}}{K! \Gamma(a)}$$

$$F_{\text{LoIW}}(x, a, b, \lambda, \vartheta) = 1 - \Upsilon e^{-(i+2k)\lambda x^{-\vartheta}} \quad (9)$$

$$\left(1 - \frac{e^{-\lambda x^{-\vartheta}} \log(1 - e^{-\lambda x^{-\vartheta}})}{b}\right)^{-(a+1)} = \sum_{k=i=0}^{\infty} \frac{(-1)^i \Gamma(a+1+k)b^{-k} d_{k,i}}{k! \Gamma(a+1)} e^{-(i+2k)\lambda x^{-\vartheta}}$$

Substituting the result of this expansion into equation (6) we get:

$$\begin{aligned} f_{\text{LoIW}}(x, a, b, \lambda, \vartheta) &= \sum_{k=i=0}^{\infty} \frac{(-1)^i \Gamma(a+1+k) a \lambda \vartheta d_{k,i}}{k! \Gamma(a+1) b^{k+1}} x^{-(\vartheta+1)} e^{-(i+2k+1)\lambda x^{-\vartheta}} \frac{e^{-\lambda x^{-\vartheta}}}{1 - e^{-\lambda x^{-\vartheta}}} \\ &\quad - \sum_{k=i=0}^{\infty} \frac{(-1)^i \Gamma(a+1+k) a \lambda \vartheta d_{k,i}}{k! \Gamma(a+1) b^{k+1}} x^{-(\vartheta+1)} e^{-(i+2k+1)\lambda x^{-\vartheta}} \log(1 - e^{-\lambda x^{-\vartheta}}) \end{aligned}$$

Now it is expanding $\frac{1}{1 - e^{-\lambda x^{-\vartheta}}}$, and $\log(1 - e^{-\lambda x^{-\vartheta}})$ respectively:

$$\begin{aligned} \frac{1}{1 - e^{-\lambda x^{-\vartheta}}} &= \sum_{z=0}^{\infty} (-1)^z [e^{-\lambda x^{-\vartheta}}]^z = \sum_{z=0}^{\infty} (-1)^z e^{-z\lambda x^{-\vartheta}} \\ \log(1 - e^{-\lambda x^{-\vartheta}}) &= \sum_{j=0}^{\infty} (-1)^j d_{1,j} (e^{-\lambda x^{-\vartheta}})^{j+1} = \sum_{j=0}^{\infty} (-1)^j d_{1,j} e^{-(j+1)\lambda x^{-\vartheta}} \end{aligned}$$

Where $d_{1,j} = \frac{1}{j} \sum_{m=1}^j \frac{[2m-j]}{m+1}$ for $j \geq 0$ and $d_{1,0} = 1$

From it we get the pdf function for the LoIW distribution in the form:

$$\begin{aligned} f_{\text{LoIW}}(x, a, b, \lambda, \vartheta) &= \sum_{k=i=z=0}^{\infty} \frac{(-1)^{i+z} \Gamma(a+1+k) a \lambda \vartheta d_{k,i}}{k! \Gamma(a+1) b^{k+1}} x^{-(\vartheta+1)} e^{-(i+2k+z+2)\lambda x^{-\vartheta}} \\ &\quad - \sum_{k=i=j=0}^{\infty} \frac{(-1)^{i+j} \Gamma(a+1+k) a \lambda \vartheta d_{k,i} d_{1,j}}{k! \Gamma(a+1) b^{k+1}} x^{-(\vartheta+1)} e^{-(i+2k+j+2)\lambda x^{-\vartheta}} \quad (10) \end{aligned}$$

3.2 Quantile function of LoIW distribution

The Quantile function of LoIW distribution by form [12]:

$$Q(u) = F^{-1}(u)$$

Where $Q(u)$ is the Quantity function $F_{LoIW}(x, a, b, \lambda, \vartheta)$ for each $u \in (0,1)$. Then the Quantile function of LoIW distribution by form:

$$Q(u) = \left(\frac{-\lambda}{\ln \left[\frac{b - \frac{b}{(1-u)^{\frac{1}{a}}}}{b - \frac{b}{(1-u)^{\frac{1}{a}}} + W_{-1} \left(b - \frac{b}{(1-u)^{\frac{1}{a}}} e^{-\left(b - \frac{b}{(1-u)^{\frac{1}{a}}} \right)} \right)} \right]} \right)^{\frac{1}{\vartheta}} \quad (11)$$

3.3 Moments

Let x be any random variable with pdf in equation (10). Then the n – th moment of the LoIW distribution is given by:

$$\mu_n = E(x^n)_{LoIW} = Z \int_0^{\infty} x^{n-(\vartheta+1)} e^{-(i+2k+z+2)\lambda x^{-\vartheta}} dx - H \int_0^{\infty} x^{n-(\vartheta+1)} e^{-(i+2k+j+2)\lambda x^{-\vartheta}} dx$$

Let

$$\begin{aligned} y &= (i+2k+z+2)\lambda x^{-\vartheta} \Rightarrow x = \frac{y^{\frac{1}{\vartheta}}}{\lambda^{\frac{1}{\vartheta}}(i+2k+z+2)^{\frac{1}{\vartheta}}} \Rightarrow dx = \frac{\frac{1}{\vartheta} y^{\frac{1}{\vartheta}-1} dy}{\lambda^{\frac{1}{\vartheta}}(i+2k+z+2)^{\frac{1}{\vartheta}}} \\ u &= (i+2k+j+2)\lambda x^{-\vartheta} \Rightarrow x = \frac{u^{\frac{1}{\vartheta}}}{\lambda^{\frac{1}{\vartheta}}(i+2k+j+2)^{\frac{1}{\vartheta}}} \Rightarrow dx = \frac{\frac{1}{\vartheta} u^{\frac{1}{\vartheta}-1} du}{\lambda^{\frac{1}{\vartheta}}(i+2k+j+2)^{\frac{1}{\vartheta}}} \\ \mu_n &= \frac{Z}{\vartheta \lambda^{\frac{n+1}{\vartheta}} (i+2k+z+2)^{\frac{n+1}{\vartheta}}} \int_0^{\infty} y^{\frac{n+2-\vartheta}{\vartheta}-1} e^{-y} dy - \frac{H}{\vartheta \lambda^{\frac{n+1}{\vartheta}} (i+2k+j+2)^{\frac{n+1}{\vartheta}}} \int_0^{\infty} u^{\frac{n+2-\vartheta}{\vartheta}-1} e^{-u} du \\ \mu_n &= \frac{\Gamma\left(\frac{n+2-\vartheta}{\vartheta}\right)}{\vartheta \lambda^{\frac{n+1}{\vartheta}}} \left[\frac{Z}{(i+2k+z+2)^{\frac{n+1}{\vartheta}}} - \frac{H}{(i+2k+j+2)^{\frac{n+1}{\vartheta}}} \right] \end{aligned} \quad (12)$$

The variance of the LoIW distribution is obtained by the following formula ($\sigma^2 = \mu_2 - \mu_1^2$). The skewness (SK) and kurtosis (KU) are defined by:

$$SK_{LoIW} = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{\frac{\Gamma\left(\frac{5-\vartheta}{\vartheta}\right)}{\vartheta \lambda^{\frac{4}{\vartheta}}} \left[\frac{Z}{(i+2k+z+2)^{\frac{4}{\vartheta}}} - \frac{H}{(i+2k+j+2)^{\frac{4}{\vartheta}}} \right]}{\left[\frac{\Gamma\left(\frac{4-\vartheta}{\vartheta}\right)}{\vartheta \lambda^{\frac{3}{\vartheta}}} \left[\frac{Z}{\vartheta \lambda^{\frac{3}{\vartheta}} (i+2k+z+2)^{\frac{3}{\vartheta}}} - \frac{H}{(i+2k+j+2)^{\frac{3}{\vartheta}}} \right] \right]^{\frac{3}{2}}} \quad (13)$$

$$KU_{LoIW} = \frac{\mu_4}{\mu_2^2} = \frac{\frac{\Gamma\left(\frac{6-\vartheta}{\vartheta}\right)}{\vartheta\lambda^{\frac{5}{\vartheta}} \left[\frac{Z}{(i+2k+z+2)^{\frac{5}{\vartheta}}} - \frac{H}{(i+2k+j+2)^{\frac{5}{\vartheta}}} \right]}}{\left[\frac{\Gamma\left(\frac{4-\vartheta}{\vartheta}\right)}{\vartheta\lambda^{\frac{3}{\vartheta}} \left[\frac{Z}{(i+2k+z+2)^{\frac{3}{\vartheta}}} - \frac{H}{(i+2k+j+2)^{\frac{3}{\vartheta}}} \right]} \right]^2} \quad (14)$$

3.4 Moment Generating Function

The moment generating function (mgf) is given by:

$$SM_t(y)_{OLE} = E(e^{yt}) = \int_{-\infty}^{\infty} e^{yt} f_{OLIW}(x, a, b, \lambda, \vartheta) dt \quad (15)$$

Used series expansion for e^{yt}

$$M_t(y)_{OLIW} = \sum_{n=0}^{\infty} \frac{y^n}{n!} E(t^n) = \sum_{n=0}^{\infty} \frac{y^n}{n!} [\mu_n] \quad (16)$$

Therefore form (12), the (mgf) of the LoIW distribution is given as flowing:

$$M_t(y)_{LoIW} = \sum_{n=0}^{\infty} \frac{y^n}{n!} \left[\frac{\Gamma\left(\frac{n+2-\vartheta}{\vartheta}\right)}{\vartheta\lambda^{\frac{n+1}{\vartheta}}} \left[\frac{Z}{(i+2k+z+2)^{\frac{n+1}{\vartheta}}} - \frac{H}{(i+2k+j+2)^{\frac{n+1}{\vartheta}}} \right] \right] \quad (17)$$

3.5 Rényi Entropy

The Rényi entropy for the LoIW distribution can be obtained:

$$I_R(c)_{LoIW} = \frac{1}{1-c} \log \int_0^{\infty} f(x)^c dt$$

Then

$$I_R(c)_{LoIW} = \frac{1}{1-c} \log \int_0^{\infty} (Zx^{-(\vartheta+1)} e^{-(i+2k+z+2)\lambda x^{-\vartheta}} - Hx^{-(\vartheta+1)} e^{-(i+2k+j+2)\lambda x^{-\vartheta}})^c dx$$

$$I_R(c)_{LoIW} = \frac{1}{1-c} \log \left[\int_0^{\infty} \sum_{m=0}^c (-1)^m \binom{c}{m} Z \cdot H x^{-c(\vartheta+1)} e^{-\lambda x^{-\vartheta}(mz+ic+2ki+cj+2c-mj)} dx \right]$$

Let

$$u = \lambda x^{-\vartheta}(mz+ic+2ki+cj+2c-mj) \Rightarrow x = \frac{u^{\frac{1}{\vartheta}}}{\lambda^{\frac{1}{\vartheta}}(mz+ic+2ki+cj+2c-mj)^{\frac{1}{\vartheta}}}$$

$$\Rightarrow dx = \frac{u^{\frac{1}{\vartheta}-1} du}{\vartheta\lambda^{\frac{1}{\vartheta}}(mz+ic+2ki+cj+2c-mj)^{\frac{1}{\vartheta}}}$$

$$I_R(c)_{LoIW} = \frac{1}{1-c} \log \left[\int_0^{\infty} \sum_{m=0}^c \frac{(-1)^m \binom{c}{m} Z \cdot H u^{\frac{1}{\vartheta}(1-c(\vartheta+1))}}{\vartheta\lambda^{\frac{1}{\vartheta}[1-c(\vartheta+1)]} (mz+ic+2ki+cj+2c-mj)^{\frac{1}{\vartheta}[1-c(\vartheta+1)]}} e^{-u} du \right]$$

$$I_R(c)_{LoIW} = \frac{1}{1-c} \log \left[\sum_{m=0}^c \frac{(-1)^m \binom{c}{m} Z \cdot H \cdot \Gamma\left(\frac{1-c(\vartheta+1)}{\vartheta}\right)}{\vartheta\lambda^{\frac{1}{\vartheta}[1-c(\vartheta+1)]} (mz+ic+2ki+cj+2c-mj)^{\frac{1}{\vartheta}[1-c(\vartheta+1)]}} \right] \quad (18)$$

3.6 Order statistics

The pdf of the j – th order statistic for a random sample of size n from a distribution function $F_{LoIW}(x, a, b, \lambda, \vartheta)$ and an associated pdf $f_{LoIW}(x, a, b, \lambda, \vartheta)$ is given by:

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} [F_{OLIW}(x, a, b, \lambda, \vartheta)]^{j+r-1} f_{OLIW}(x, a, b, \lambda, \vartheta) \quad (19)$$

Where $F_{OLIW}(x, a, b, \lambda, \vartheta)$ CDF of LoIW dist. and $f_{OLIW}(x, a, b, \lambda, \vartheta)$ is pdf of LoIW dist. while $k = \frac{n!}{(i-1)!(n-i)!}$. However, the following

is the pdf of the j -th order statistics for a random sample of size n drawn from the LoIW dist. :

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} a \lambda \vartheta x^{-(\vartheta+1)} e^{-\lambda x^{-\vartheta}} \frac{\left[1 - \left(1 - \frac{e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})}{b}\right)^{-a}\right]^{j+r}}{b - e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})} \times \left[\frac{e^{-\lambda x^{-\vartheta}}}{1 - e^{-\lambda x^{-\vartheta}}} - \log(1 - e^{-\lambda x^{-\vartheta}})\right] \quad (20)$$

So, the $f_{j:n}(t)$ of minimum order statistics is obtained by substituting $j = 1$ in Equation (20) to have:

$$f_{1:n}(x) = \sum_{r=0}^{n-1} k(-1)^r \binom{n-1}{r} a \lambda \vartheta x^{-(\vartheta+1)} e^{-\lambda x^{-\vartheta}} \frac{\left[1 - \left(1 - \frac{e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})}{b}\right)^{-a}\right]^{1+r}}{b - e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})} \times \left[\frac{e^{-\lambda x^{-\vartheta}}}{1 - e^{-\lambda x^{-\vartheta}}} - \log(1 - e^{-\lambda x^{-\vartheta}})\right] \quad (21)$$

While the corresponding $f_{j:n}(t)$ of maximum order statistics is obtained by making the substitution of $j = n$ in Equation (20) as:

$$f_{n:n}(x) = k a \lambda \vartheta x^{-(\vartheta+1)} e^{-\lambda x^{-\vartheta}} \frac{\left[1 - \left(1 - \frac{e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})}{b}\right)^{-a}\right]^{n+r}}{b - e^{-\lambda x^{-\vartheta}} \cdot \log(1 - e^{-\lambda x^{-\vartheta}})} \left[\frac{e^{-\lambda x^{-\vartheta}}}{1 - e^{-\lambda x^{-\vartheta}}} - \log(1 - e^{-\lambda x^{-\vartheta}})\right] \quad (22)$$

4. Estimation

The parameters of the LoIW dist. are estimated using the maximum likelihood estimation approach. The log-likelihood

function is derived for a random sample $x_1, x_2, \dots, x_n$ distributed in accordance with the pdf of the LoIW dist..

$$L(\theta, x_i) = \prod_{i=1}^n f_{OLIW}(x_i, \delta, b, \lambda, \vartheta)$$

$$(\theta, x_i) = \prod_{i=1}^n \frac{a \lambda \vartheta}{b} x_i^{-(\vartheta+1)} e^{-\lambda x_i^{-\vartheta}} \left(1 - \frac{e^{-\lambda x_i^{-\vartheta}} \cdot \log(1 - e^{-\lambda x_i^{-\vartheta}})}{b}\right)^{-(a+1)} \left[\frac{e^{-\lambda x_i^{-\vartheta}}}{1 - e^{-\lambda x_i^{-\vartheta}}} - \log(1 - e^{-\lambda x_i^{-\vartheta}})\right]$$

The log-likelihood function L is obtained as:

$$\begin{aligned} &= n \log a + n \log \lambda + n \log \vartheta - n \log b - (\vartheta + 1) \sum_{i=1}^n \log x_i - \lambda \sum_{i=1}^n x_i^{-\vartheta} \\ &\quad - (a + 1) \sum_{i=1}^n \log \left(1 - \frac{e^{-\lambda x_i^{-\vartheta}} \cdot \log(1 - e^{-\lambda x_i^{-\vartheta}})}{b}\right) \\ &\quad + \sum_{i=1}^n \log \left(\frac{e^{-\lambda x_i^{-\vartheta}}}{1 - e^{-\lambda x_i^{-\vartheta}}} - \log(1 - e^{-\lambda x_i^{-\vartheta}})\right) \\ &\quad \frac{\partial L}{\partial a} = \frac{n}{a} - \sum_{i=1}^n \log \left(1 - \frac{e^{-\lambda x_i^{-\vartheta}} \cdot \log(1 - e^{-\lambda x_i^{-\vartheta}})}{b}\right) \end{aligned} \quad (23)$$

$$\begin{aligned}
\frac{\partial L}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n x_i^{-\theta} - (a+1) \sum_{i=1}^n \frac{x_i^{-\theta} e^{-2\lambda x_i^{-\theta}}}{1 - e^{-\lambda x_i^{-\theta}}} + x_i^{-\theta} e^{-\lambda x_i^{-\theta}} \log(1 - e^{-\lambda x_i^{-\theta}}) \\
&\quad - \frac{(1 - e^{-\lambda x_i^{-\theta}}) x_i^{-\theta} e^{-\lambda x_i^{-\theta}} - x_i^{-\theta} e^{-2\lambda x_i^{-\theta}}}{(1 - e^{-\lambda x_i^{-\theta}})^2} - \frac{x_i^{-\theta} e^{-2\lambda x_i^{-\theta}}}{1 - e^{-\lambda x_i^{-\theta}}} \\
&\quad + \sum_{i=1}^n \frac{\frac{e^{-\lambda x_i^{-\theta}}}{1 - e^{-\lambda x_i^{-\theta}}} - \log(1 - e^{-\lambda x_i^{-\theta}})}{1 - e^{-\lambda x_i^{-\theta}}} \\
\frac{\partial L}{\partial b} &= -\frac{n}{b} - (a+1) \sum_{i=1}^n \frac{e^{-\lambda x_i^{-\theta}} \cdot \log(1 - e^{-\lambda x_i^{-\theta}})}{b^2 - b e^{-\lambda x_i^{-\theta}} \cdot \log(1 - e^{-\lambda x_i^{-\theta}})} \\
&\quad - \frac{\lambda x_i^{-\theta} e^{-\lambda x_i^{-\theta}} \log x_i [\log(1 - e^{-\lambda x_i^{-\theta}}) - e^{-\lambda x_i^{-\theta}}]}{b - e^{-\lambda x_i^{-\theta}} \cdot \log(1 - e^{-\lambda x_i^{-\theta}})} \\
\frac{\partial L}{\partial \theta} &= \frac{n}{\theta} - \sum_{i=1}^n \log x_i + \lambda \sum_{i=1}^n x_i^{-\theta} \log x_i + (a+1) \sum_{i=1}^n \frac{\frac{1 - e^{-\lambda x_i^{-\theta}}}{b - e^{-\lambda x_i^{-\theta}} \cdot \log(1 - e^{-\lambda x_i^{-\theta}})}}{1 - e^{-\lambda x_i^{-\theta}}} \\
&\quad - \frac{(1 - e^{-\lambda x_i^{-\theta}}) \lambda x_i^{-\theta} e^{-\lambda x_i^{-\theta}} \log x_i + e^{-2\lambda x_i^{-\theta}} \lambda x_i^{-\theta} \log x_i}{(1 - e^{-\lambda x_i^{-\theta}})^2} + \frac{\lambda x_i^{-\theta} e^{-\lambda x_i^{-\theta}} \log x_i}{1 - e^{-\lambda x_i^{-\theta}}} \\
&\quad + \sum_{i=1}^n \frac{\frac{e^{-\lambda x_i^{-\theta}}}{1 - e^{-\lambda x_i^{-\theta}}} - \log(1 - e^{-\lambda x_i^{-\theta}})}{1 - e^{-\lambda x_i^{-\theta}}}
\end{aligned}$$

The solution of the non-linear equations of $\frac{\partial L}{\partial a} = 0$, $\frac{\partial L}{\partial \lambda} = 0$, $\frac{\partial L}{\partial b} = 0$, and $\frac{\partial L}{\partial \theta} = 0$ results to the ML estimates of parameters a, λ, b, θ respectively. The solution could not be obtained analytically except by numerical methods using software like R, MAPLE, SAS and so on.

5. Application

This section includes two datasets that highlight the capabilities of the LoIW models for modeling income data. The potential of the supplied model is measured by comparing its performance to several other introduced models, like :Gompertz Inverse Weibull (GoIW) distribution [13], Truncated Exponential Inverse Weibull (TEEIW) distribution [14], Beta Inverse Weibull (BeIW) distribution [15], Kumaraswamy Inverse Weibull (KuIW) distribution [16], Exponential Generalized Inverse Weibull (EGIW) distribution [17], and Inverse Weibull (IW) distribution [18].

The first dataset: the data include 30 observation of march precipitation in Minneapolis/st. Paul (measured in inches). [19] The data are (0.77 , 1.74 , 0.81 , 1.20 , 1.95 , 1.20 ,0.47 , 1.43 , 3.37 , 2.20 , 3.00 , 3.09 , 1.51 , 2.10 , 0.52 , 1.62 , 1.31 , 0.32, 0.59 , 0.81 , 2.81 , 1.87 , 1.18 ,1.35 , 4.75 , 2.48 , 0.96 , 1.89 , 0.90 , 2.05)

The second dataset : Kundu and Raqab [20] investigated 74 observation of gauge lengths of 20mm. the data are :

1.312, 1.314 ,1.479, 1.552 ,1.700, 1.803 ,1.861, 1.865 ,1.944, 1.958 ,1.966, 1.997 ,2.006 ,2.021, 2.027,2.055 ,2.063 ,2.098 ,2.140 ,2.179 ,2.224 ,2.240, 2.253 ,2.270 ,2.272 ,2.274 ,2.301 ,2.301 ,2.359 ,2.382,2.382 ,2.426 ,2.434,2.435 ,2.478 ,2.490 ,2.511 ,2.514 ,2.535 ,2.554 ,2.566 ,2.570 ,2.586 ,2.629, 2.633 ,2.642 ,2.648 ,2.684 ,2.697 ,2.726 ,2.770 ,2.773 ,2.800 ,2.809 ,2.818, 2.821 ,2.848 ,2.880,2.809 ,2.818 ,2.821 ,2.848 ,2.880, 2.954 ,3.012, 3.067, 3.084 ,3.090, 3.096 ,3.128, 3.233 ,3.433, 3.585 ,3.585

Table 1. Goodness of fit statistics for Data I.

Distribution	MLEs	-LL	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
LoIW	1.1739 0.0475 3.6006	38.06	84.125	85.725	89.730	85.918	0.0145	0.1045	0.0620	0.9998

	0.2974									
TEEIW	0.1995 0.4289 2.4928 1.5106	41.90	91.819	93.419	97.424	93.612	0.1252	0.7670	0.1508	0.5024
BeIW	0.3820 6.6254 5.2586 0.9185	38.48	84.974	86.574	90.578	86.767	0.0326	0.2092	0.0985	0.9325
KuIW	1.7416 5.1539 1.5385 0.8149	39.08	86.176	87.776	91.781	87.969	0.0492	0.3090	0.1062	0.8873
EGIW	5.4692 0.5102 3.9462 0.9427	38.76	85.542	87.142	91.147	87.335	0.0409	0.2592	0.1087	0.8699
IW	1.0252 1.5496	41.91	87.834	88.278	90.636	88.730	0.1260	0.7721	0.1523	0.4892

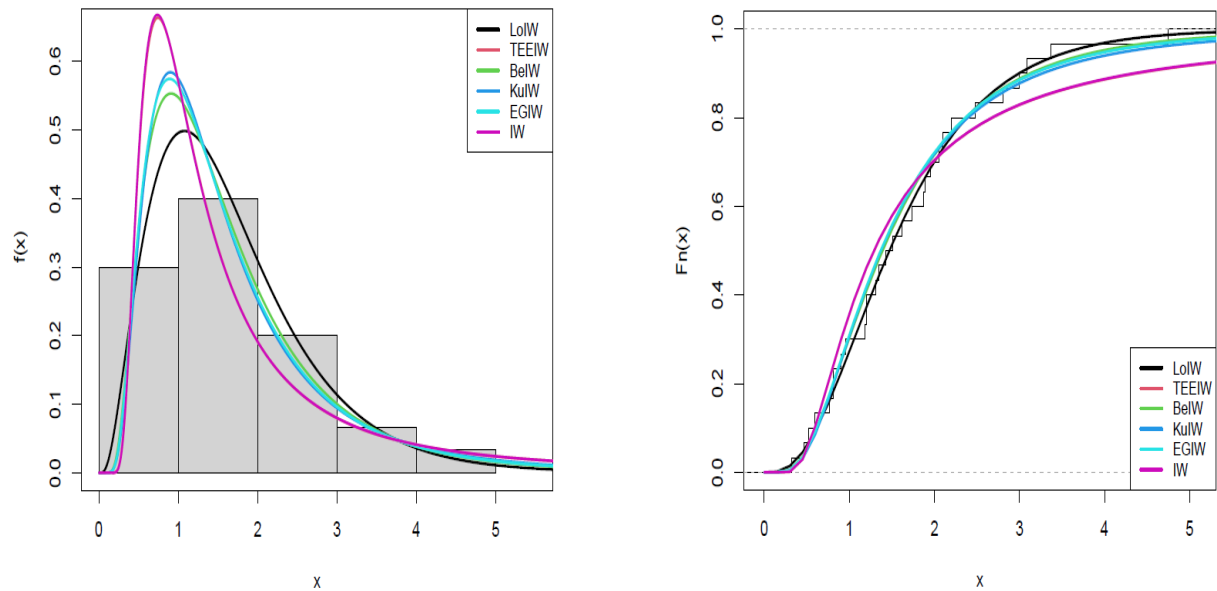


Figure 2. Empirical density and CDF for Data I.

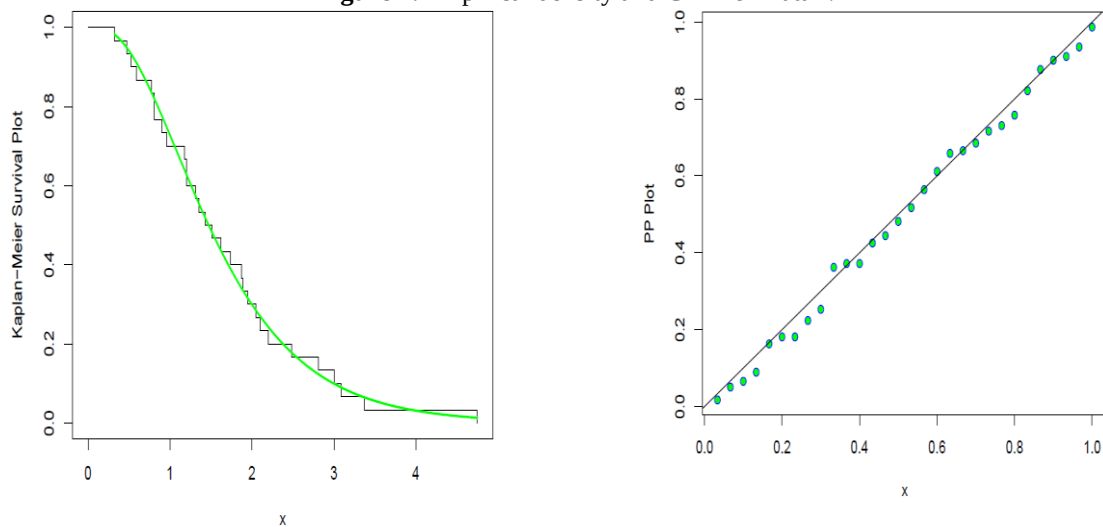
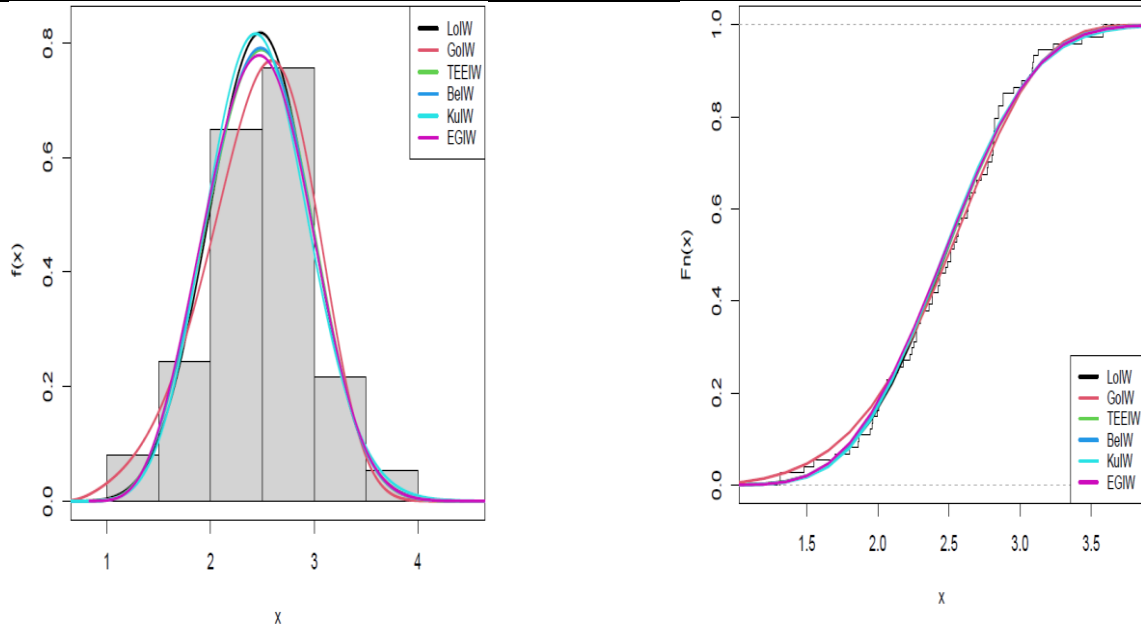


Figure 3. Kaplan and PP Plot for the Data I.

Table 2. Goodness of fit statistics for data II.

Distribution	MLEs	-LL	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
LoIW	5.9697 0.0464 7.7134 0.7389	51.34	110.69	111.27	119.91	114.37	0.0372	0.2674	0.0537	0.9829
GoIW	0.0321 2.2321 2.1481 2.6389	51.84	111.69	112.27	120.91	115.37	0.0285	0.2542	0.0696	0.8658
TEEIW	2.4797 0.4666 2.4997 1.4743	51.45	110.91	111.49	120.13	114.59	0.0395	0.2988	0.0573	0.9679
BeIW	0.4272 4.3432 6.8723 1.5526	51.51	111.03	111.61	120.24	114.70	0.0409	0.3072	0.0538	0.9827
KuIW	3.9736 1.7437 0.4418 1.0371	51.93	111.86	112.44	121.08	115.54	0.0544	0.3708	0.0594	0.9561
EGIW	6.9349 0.4182 2.3959 1.6622	51.65	111.31	111.89	120.53	114.99	0.0447	0.3335	0.0552	0.9775

**Figure 4.** Empirical density and CDF for Data II.

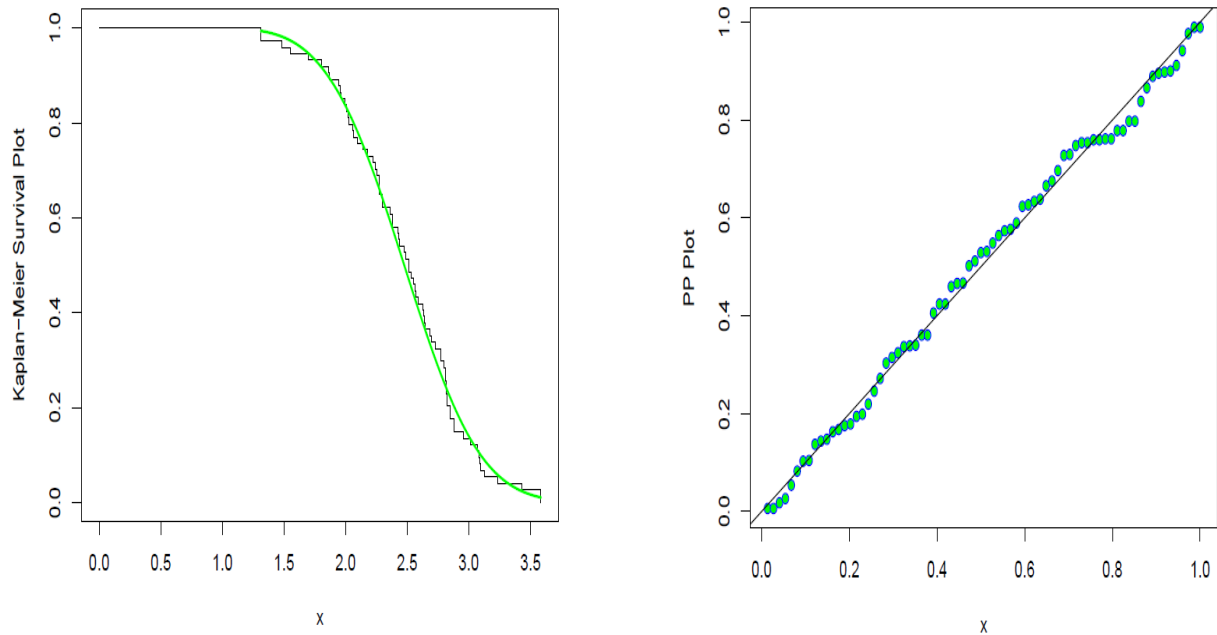


Figure 5. Kaplan and PP Plot for the Data II.

6. Results and Discussion

Tables 2 and 3 for Data I and Data II, respectively, display parameter estimates for the LoIW distribution and its sub models, as well as LL and statistics such as AIC, CAIC, BIC, HQIC, W, A, KS statistic value, and highest P-Value. Among all the distribution tested, the LoIW distribution yields the lowest values for these statistic value. As a result, it can be concluded that the LoIW distribution provides the best fit for both Data, and is a highly competitive model when compared to other distribution.

This finding is supported by examining the dataset's histogram and the presented fitted densities of the LoIW distribution and its sub models, as depicted in Figures 2-5. These visual representations demonstrate the LoIW distribution's superior ability to accurately capture data features.

7. Conclusions

In this study, a four-parameter distribution called Odd lomax Inverse Weibull (LoIW) distribution was proposed, and some characteristics of the proposed distribution were applied. The proposed distribution had very high flexibility. This flexibility gave the new distribution the ability to model the practical application of data on two different

groups. A comparison of the proposed model was also made with the models Gompertz Inverse Weibull, Truncated Exponential Inverse Weibull, Beta Inverse Weibull, Kumaraswamy Inverse Weibull, and Exponential Generalized Inverse Weibull and Inverse Weibull, which gave lower standard values as it had higher accuracy and stronger fit.

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