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Modified Weibull-Fréchet Distribution Properties with Application

Rihab Ahmed Abed[1], Mundher A. Khaleel[2], Nooruldeen A. Noori[3]

[1] Statistics Departments, College Management and Economics, University of Basra, 61001 Basra, Iraq
[2]Mathematics Departments, College of Computer Science and Mathematics, Tikrit University, 34001 Baghdad, Iraq
[3]Anbar Education Directorate, Anbar, Iraq, 31008 Anbar, Iraq

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ABSTRACT

In this paper, develop new statistical distribution From within the continuous distributions based on the Hybrid Weibull-G family and the Frechet distribution, called the Hybrid Weibull Frechet. This development comes by adding two new parameters to the basic distributions in order to give more flexibility and suitability to the basic Frechet distribution for reading and modeling real-world data. Also present the basic functions of the new distribution (pdf, CDF). In addition, the paper will present an expansion of these functions and an expansion of the powers on which a number of statistical properties of the two proposed distributions will be built, such as the moments function, Incomplete Moments, Probability Weighted Moments, the survival function, the quantity function, and four types of entropy and disparity measures that represent a small number of... Numerous mathematical and statistical features of the developed distribution. A parameters were also estimated using the MLE. In order to obtain a distribution characterized by high flexibility to match different types of real data, a simulation was conducted to demonstrate the efficiency of estimating the unknown parameters of the new distribution using the maximum likelihood method. In addition, each distribution was tested on two types of real data of different sizes. The proposed new distribution proved highly flexible for the data when compared to other distributions through the use of some statistical standards such as AIC, BIC, CAIC, and HQIC. The results obtained showed the extent of flexibility and accuracy that supports what was discussed in the theoretical aspect.

1. Introduction

Over the past few decades, a variety of classical distributions have emerged and have been widely used to model data in a variety of fields, including engineering, innovative, environmental, medical, biological and demographic studies, economics, and insurance. However, there is clear demand for extended versions of these distributions in a number of practical areas, including life analysis and insurance. For this reason, there are a number of techniques for creating new families of studied distributions. There have been some attempts to create new families of probability distributions that extend known families of distributions while providing excellent practicability for data modelling. When working with traditional statistical distributions, such as normal, exponential, or other distributions, we may encounter some problems in specific applications. One such problem is our inability to effectively account for abnormal or asymmetric data. In many practical cases, data analysis with unusually shaped distributions is required. For example, we may encounter data that shows extreme left or extreme right distributions, where values are highly concentrated at one extreme. Or we may encounter unevenly distributed data restricted to specific values, such as temporal data, which requires dealing with recent distributions. In order to solve this problem, new families of

Corresponding Author: E-**mail** address: Nooruldeen.a.noori35508@st.tu.edu.iq

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statistical distributions are created with different representativeness or distribution. Properties that fit the non-traditional data must be developed and used. Therefore, researchers have recently focused greatly on generalizing and studying classical distributions, and investigating their flexibility and applicability. It was found that these new composite distributions, in their improved form, are suitable for applications, compared to the usual classical survival models, which are obtained by introducing one or more additional shape parameters to the original distribution.

One of these attempts to create families of continuous distributions was presented by researcher Alzaatreh in 2013, where he developed a new method to find the family CDF function in the form [1]:

$$F(x;\zeta)=\int_{0}^{\mathcal{F}(G(x;\zeta))} t(u)\,du \qquad (1)$$

With three conditions for function $\mathcal{F}(G(x;\zeta))$, where $G(x;\zeta)$ is CDF of a new family, $t(u)$ is pdf random variable $u\in[a,b]$.

According to the above principle, many researchers have developed functions F(G(x;ζ)), and there are many examples of these functions, including, for example: in 2012 Alexander introduced the Mc-G family and established a new F(G(x;ζ)) function [2], also Risti’c introduced Gamma-G [3], Alzaghal introduced Exponented T-X [4], Torabi in 2014 introduced The logistic-uniform distribution [5], Tahir introduced OGE-G family [6], Ahmad introduced Weibull-X [7], and Nouri and others presented a new integration limit that satisfies the conditions [8]:

a. $-\,log\big(1-G(x;\zeta)\big)^{G(x;\zeta)}\in[a,b],-\infty<a<b<\infty$

b. $-\,log\big(1-G(x;\zeta)\big)^{G(x;\zeta)}$ is differentiable and monotonically non-decreasing (1)

c. $-\,log\big(1-G(x;\zeta)\big)^{G(x;\zeta)}\rightarrow 0, as\ x\rightarrow 0,$ and $-\,log\big(1-G(x;\zeta)\big)^{G(x;\zeta)}\rightarrow 1, as\ x\rightarrow\infty$

Using the terms mentioned above, quite a few families of continuous distributions were found, including, for example: Gompertz-G family by [9], MOTL-G family by [10], APRAY-G by [11], TIIEHL-Gom-TL-G by [12], Topp–Leone family [13], and Marshal-Olkin transformation family [14].

Depending on the integration term [8] and the pdf function and the Weibull distribution, the CDF function of the HWG family with parameters k and c is obtained in the form [15]:

$$F_{HWG}(x,k,c,\zeta)=1-e^{\left(-k\left[-G(x;\zeta).log\left(1-G(x;\zeta)\right)\right]^{c}\right)} \qquad (2)$$

While pdf function of the HWG family is obtained in the form:

$$f_{HWG}(x,k,c,\zeta)=kc\ g(x;\zeta)\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}-log\big(1-G(x;\zeta)\big)\right] \times\left[-G(x;\zeta).\,log\big(1-G(x;\zeta)\big)\right]^{c-1}e^{\left(-k\left[-G(x;\zeta).log\left(1-G(x;\zeta)\right)\right]^{c}\right)} \qquad (3)$$

Where $G(x;\zeta)$ is baseline distribution, $x>0$, and $k,c>0$

The study aims to find a new continuous statistical distribution with four parameters, while finding a number of statistical characteristics of the developed distribution to obtain a flexible distribution with real data, with a practical application on two different types of data to confirm the flexibility and suitability of the developed distribution.

The study included five sections, where the first section included the new developed distribution, while the second section included proving many of the statistical properties of the distribution while drawing the basic distribution functions with different values of the parameters, while the third section included estimating the distribution parameters using the maximum likelihood method, and in the fourth section a simulation was presented using the Monte Carlo method, was estimates parameters using the maximum likelihood method. The fifth and final section includes a practical application to two types of real data and comparison with some known and new distributions.

2. Hybrid Weibull Frechet (HWFr) distribution

The CDF and pdf functions of the Frechet distribution with two parameters a and b are respectively given by the formulas:

$$G(x,a,b) = e^{-\left(\frac{b^a}{x^a}\right)}, \quad x > 0 \qquad (4)$$

$$g(x,a,b) = ab^a x^{-(a+1)} e^{-\left(\frac{b^a}{x^a}\right)}, \quad x > 0 \qquad (5)$$

The CDF distribution function for the four-parameter Hybrid Weibull Frechet (HWFR) distribution may be found by substituting Eq(5) into Eq(3), and the pdf distribution can be obtained by substituting Eq(6) into Eq(4). Consequently, we get the following functions for the CDF and pdf HWFr distributions, respectively:

$$F_{HWFr}(x) = 1 - e^{\left(-k\left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^c\right)}, \quad x,k,c,a,b > 0 \qquad (6)$$

$$f_{HWFr}(x) = kc\, ab^a x^{-(a+1)} e^{-\left(\frac{b^a}{x^a}\right)} \left[\frac{e^{-\left(\frac{b^a}{x^a}\right)}}{1-e^{-\left(\frac{b^a}{x^a}\right)}} - log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right] \qquad (7)$$

$$\times \left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^{c-1} e^{\left(-k\left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^c\right)}$$

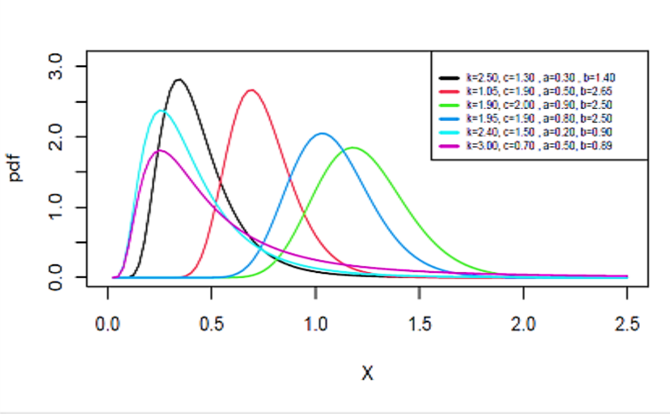


Figure 1. pdf of HWFr distribution

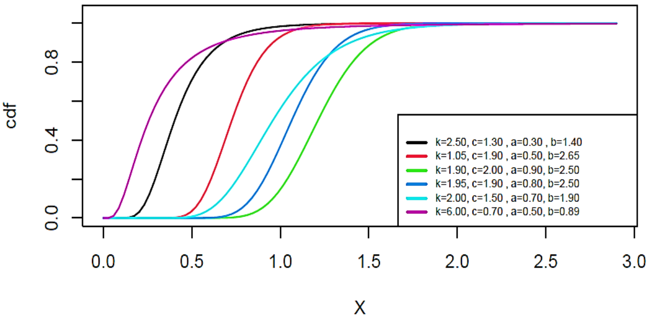


Figure 2. CDF of HWFr distribution

The curves of the CDF function shown in the previous figure follow the rules set out in the CDF function standard and are in the value range of 0 to 1. At different heights, the picture above shows that some HWFr distribution pdf function shapes are skewed to the right and others are skewed to the left. This means that the new distribution can understand a lot of different types of data.

Figure 1 shows that classical Fréchet distribution has a single shape (a single curve with a heavy right tail), while HWFr is more flexible due to addition of two new Weibull parameters (k, c), resulting in multiple shapes (such as right- or left skewed curves or multiple peaks) this makes HWFr more capable of representing complex real-world data. While figure 2 shows classical Fréchet CDF based on only two parameters (a, b) , the HWFr CDF is affected by parameters, making it more accurate in describing data with asymmetric or extreme characteristics.

3. Mathematical Properties for HWFr distribution

3.1 Survival, hazard, and odd functions

The survival function is defined as the probability that the system will not fail after a period of time. The survival function for the HWFr distribution is obtained by the following relationship [17]:

$$S_{HWFr}(x) = e^{\left(-k\left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^c\right)} \tag{8}$$

The hazard function has great importance in many fields, such as various sciences and engineering and its departments, especially with regard to life issues. Therefore, many researchers have focused their attention on the hazard function and finding statistical distributions with different shapes for that function, and therefore it can be obtained for the HWFr distribution in the form [18]:

$$\begin{aligned} h(x)_{HWFr} = kc\, ab^a x^{-(a+1)} e^{-\left(\frac{b^a}{x^a}\right)} &\left[\frac{e^{-\left(\frac{b^a}{x^a}\right)}}{1-e^{-\left(\frac{b^a}{x^a}\right)}} - log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right] \\ &\times \left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^{c-1} \end{aligned} \tag{9}$$

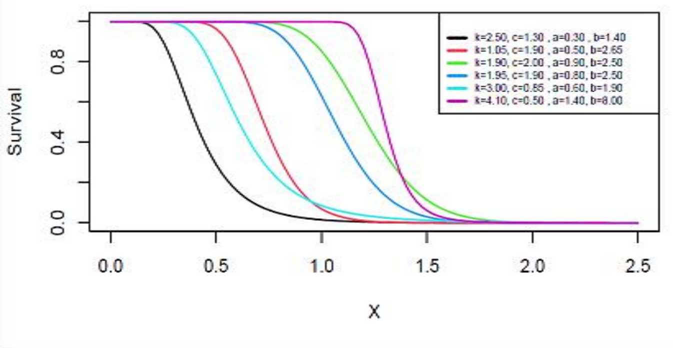


Figure 3. Survival functions for HWFr distribution

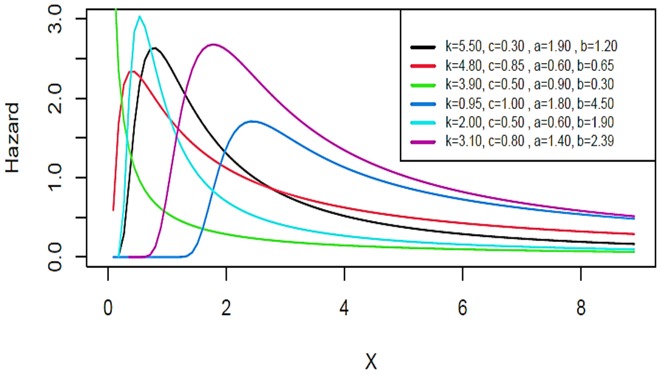


Figure 4. Hazard functions for HWFr distribution

Figure 3 illustrates the probability of values remaining above a specified values (used in survival analysis). For HWFr, the probability of values decreasing at different rates depending on the parameter values, while for classic Fréchet the decline is relatively constant. Figure 4 measures the instantaneous failure rate. Note the classic Fréchet has a constant or increasing hazard function, while HWFr can have more varied shapes (such as an inverted it suitable for data with different failure patterns.

The odd function for the HWFr distribution can be obtained in the form:

$$O_{HWFr}(x)=\left[\frac{1-e^{\left(-k\left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^{c}\right)}}{c\,ab^a x^{-(a+1)}e^{-\left(\frac{b^a}{x^a}\right)}\left[\frac{e^{-\left(\frac{b^a}{x^a}\right)}}{1-e^{-\left(\frac{b^a}{x^a}\right)}}-log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]}\right] \tag{10}$$

$$\times\left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^{1-c}$$

3.2 Useful representations pdf and cdf of HWFr distribution

The objective of this part is to enhance and broaden the PDF function of the proposed distribution HWFr. This involves representing Eq(7) using both exponential function expansion and binomial expansion. This is done in consideration of the role played by the CDF distribution function, which is as follows form:

$$F_{HWFr}(x,k,c,a,b)=1-\mathrm{E}e^{-\frac{b^a}{x^a}(l+pc)} \tag{11}$$

Where

$$\mathrm{E}=\sum_{p=0}^{\infty}\sum_{u=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{p(c+1)+u+l}}{p!}k^p\binom{u+cp}{l}$$

Regarding the extension of the $(\mathrm{CDF})^{\mathrm{z}}$ indicated by the equation below:

$$F_{HWFr}^{z}(x)=\left(1-e^{\left(-k\left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^{c}\right)}\right)^{z} \tag{12}$$

Using the expansions used to expand the CDF function, the $(\mathrm{CDF})^{\mathrm{z}}$ function is obtained in the form:

$$F_{HWFr}^{z}(x,k,c,a,b)=\mathrm{K}e^{-\frac{b^a}{x^a}(ic+l)} \tag{13}$$

where

$$\mathrm{K}=\sum_{i=0}^{\infty}\sum_{u=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i(c+1)+u+l}}{i!}z^i k^i\,d_{ic,u}\binom{u+ic}{l}$$

To enlarge the probability density function (pdf) of the HWFr distribution, we can utilise Eq(8) in the following manner: By employing the binomial series expansion and logarithm expansion. Then the last form for pdf of HWFr distribution get it using the formula:

$$f_{HWFr}(x)=x^{-(a+1)}\left[\mathrm{F}e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}}-\mathrm{M}e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}}\right] \tag{14}$$

where

$$\mathrm{F}=\sum_{p=0}^{\infty}\sum_{u=0}^{\infty}\sum_{l=0}^{\infty}\sum_{v=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{p(c+1)+u+l+v+c-1}}{j*p!}d_{cp,u}d_{c-1,v}k^{p+1}c\;ab^{a}\binom{u+cp}{l}$$

and

$$\mathrm{M}=\sum_{p=0}^{\infty}\sum_{u=0}^{\infty}\sum_{l=0}^{\infty}\sum_{v=0}^{\infty}\sum_{z=0}^{\infty}\frac{(-1)^{p(c+1)+u+l+v+c+z-1}}{p!}d_{cp,u}d_{c-1,v}d_{1,z}k^{p+1}cab^{a}\binom{u+cp}{l}$$

Using the expansions used to expand the pdf function, the f_{HWFr}^{δ} function is obtained by using Eq(15) in the form:

$$f_{HWFr}^{\delta}(x,k,c,a,b)=\left(x^{-(a+1)}\left[\mathrm{F}e^{-(l+pc+v+2c+j)\frac{b^{a}}{x^{a}}}-\mathrm{M}e^{-(l+pc+v+2c+z)\frac{b^{a}}{x^{a}}}\right]\right)^{\delta}$$

Finally we get:

$$f_{HWFr}^{\delta}(x)=x^{-\delta(a+1)}\sum_{m=0}^{z}(-1)^{m}\binom{z}{m}\mathrm{F}.\mathrm{M}\,e^{-(\delta l+\delta pc+\delta v+\delta 2c+\delta z-mz)\frac{b^{a}}{x^{a}}} \tag{15}$$

3.3 Quantile function

The Quantile function is the inverse of the CDF function in Eq(7). It is employed to determine the median, skewness, and kurtosis for distributions with significant skewness or lacking moments. Additionally, it enables the generation of random numbers for data in simulation studies. It is known as the equation:

$$x=b\left[-\ln\left(\frac{\sqrt[c]{\frac{-\ln(1-u)}{k}}}{\sqrt[c]{\frac{-\ln(1-u)}{k}}+W_{-1}\left(-\sqrt[c]{\frac{-\ln(1-u)}{k}}\,e^{\sqrt[c]{\frac{-\ln(1-u)}{k}}}\right)}\right)\right]^{-\frac{1}{a}} \tag{16}$$

$$x=b\left[-\ln\left(\frac{\sqrt[c]{\frac{-\ln(1-u)}{k}}}{\sqrt[c]{\frac{-\ln(1-u)}{k}}+W_{-1}\left(-\sqrt[c]{\frac{-\ln(1-u)}{k}}\,e^{\sqrt[c]{\frac{-\ln(1-u)}{k}}}\right)}\right)\right]^{-\frac{1}{a}} \tag{17}$$

Table.1 explains the Quantiles for selected parameter values of the HWFr distribution.

u	(k,c,b,a)				
	(1.3,0.7,1.2,0.4)	(2,2.3,0.7,1.3)	(1.3,0.3,1.3,0.7)	(0.7,3,1.8,0.3)	(1.1,4.3,1.1,1.2)
0.1	0.2622	0.8437	0.1678	15.5798	1.9901
0.2	0.5846	0.9750	0.2771	29.9030	2.2366
0.3	1.1142	1.0800	0.4256	47.2891	2.4210
0.4	2.0474	1.1768	0.6575	69.3253	2.5821
0.5	3.8415	1.2733	1.0764	98.3356	2.7349
0.6	7.7644	1.3759	2.0105	138.4063	2.8894
0.7	18.3409	1.4926	5.0665	197.8336	3.0564
0.8	60.5648	1.6388	30.9003	297.3208	3.2535
0.9	519.0596	1.8602	19487.3535	513.8483	3.5295

The results from table 1 show that the Quantile values increase with increasing value of the parameter u (from 0.1 to 0.9), reflecting the behaviour of the distribution as the parameters of the distribution as the parameters change. When the parameters are ($k=1.3$, $c=0.7$, $b=1.2$, $a=0.4$), the quantile value at $u=0.5$ the median is 3.8415, while in the classical Fréchet it will be lower due to lack of elasticity.

The Median of HWFr distribution can be finding if we but $u=0.5$ in equation (17) we get the form:

$$Median=b\left[-\ln\left(\frac{\sqrt[c]{\frac{-\ln(0.5)}{k}}}{\sqrt[c]{\frac{-\ln(0.5)}{k}}+W_{-1}\left(-\sqrt[c]{\frac{-\ln(0.5)}{k}}\,e^{\sqrt[c]{\frac{-\ln(0.5)}{k}}}\right)}\right)\right]^{-\frac{1}{a}} \tag{18}$$

3.4 Moments

Moments play a crucial role in determining the mean, variance, skewness, and kurtosis of a probability distribution. The n^{th} moments of the HWFr distribution can be obtained through the equation [18],[19]:

$$\mu_n = E(x^n)_{OLoG} = \int_0^\infty x^n f_{HWFr}(x)dx$$

By substitution equation(15) in above equation we get:

$$\mu_n = E(x^n)_{OLoG} = \int_0^\infty x^{n-a-1}\left[\mathrm{F}e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} - \mathrm{M}e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}}\right]dx$$

$$\mu_n = E(x^n)_{OLoG} = \int_0^\infty \mathrm{F}x^{n-a-1}e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}}dx - \int_0^\infty \mathrm{M}x^{n-a-1}e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}}dx$$

Let $\beta = (l+pc+v+2c+j)\frac{b^a}{x^a} \Longrightarrow x^a = \frac{b^a(l+pc+v+2c+j)}{\beta} \Longrightarrow x = \frac{b(l+pc+v+2c+j)^{\frac{1}{a}}}{\beta^{\frac{1}{a}}} \Longrightarrow dx = -\frac{b}{a}(l+pc+v+2c+j)^{\frac{1}{a}}\beta^{-\frac{(1+a)}{a}}\,d\beta$

Let $\tau = (l+pc+v+2c+z)\frac{b^a}{x^a} \Longrightarrow x^a = \frac{b^a(l+pc+v+2c+z)}{\tau} \Longrightarrow x = \frac{b(l+pc+v+2c+z)^{\frac{1}{a}}}{\tau^{\frac{1}{a}}} \Longrightarrow dx = -\frac{b}{a}(l+pc+v+2c+z)^{\frac{1}{a}}\tau^{-\frac{(1+a)}{a}}\,d\tau$

$$\mu_n = \frac{-\mathrm{F}(l+pc+v+2c+j)^{\frac{n}{a}-1}}{a}\int_0^\infty \beta^{-\frac{n}{a}}e^{-\beta}\,d\beta + \frac{\mathrm{M}(l+pc+v+2c+z)^{\frac{n}{a}-1}}{a}\int_0^\infty \tau^{-\frac{n}{a}}e^{-\tau}\,d\tau$$

$$\mu_n = \frac{-\mathrm{F}(l+pc+v+2c+j)^{\frac{n}{a}-1}}{a}\Gamma\left(-\frac{n}{a}+1\right) + \frac{\mathrm{M}(l+pc+v+2c+z)^{\frac{n}{a}-1}}{a}\Gamma\left(-\frac{n}{a}+1\right)$$

$$\mu_n = \frac{b^{n-a}\Gamma\left(-\frac{n}{a}+1\right)}{a}\left[-\mathrm{F}(l+pc+v+2c+j)^{\frac{n}{a}-1} + \mathrm{M}(l+pc+v+2c+z)^{\frac{n}{a}-1}\right] \tag{19}$$

The variance of the HWFr distribution is obtained by the following formula($\sigma^2 = \mu_2 - \mu_1^2$), and by using Eq(19)

$$\sigma^2 = \frac{b^{2-a}\Gamma\left(-\frac{2}{a}+1\right)}{a}\left[-\mathrm{F}(l+pc+v+2c+j)^{\frac{2}{a}-1} + \mathrm{M}(l+pc+v+2c+z)^{\frac{2}{a}-1}\right] - \left(\frac{b^{1-a}\Gamma\left(-\frac{1}{a}+1\right)}{a}\left[-\mathrm{F}(l+pc+v+2c+j)^{\frac{1}{a}-1} + \mathrm{M}(l+pc+v+2c+z)^{\frac{1}{a}-1}\right]\right)^2 \tag{20}$$

The skewness (SK) and kurtosis (KU) of HWFr distribution are defined by [22],[23]:

$$SK = \frac{\frac{b^{3-a}\Gamma\left(-\frac{3}{a}+1\right)}{a}\left[-\mathrm{F}(l+pc+v+2c+j)^{\frac{3}{a}-1}+\mathrm{M}(l+pc+v+2c+z)^{\frac{3}{a}-1}\right]}{\left(\frac{b^{2-a}\Gamma\left(-\frac{2}{a}+1\right)}{a}\left[-\mathrm{F}(l+pc+v+2c+j)^{\frac{2}{a}-1}+\mathrm{M}(l+pc+v+2c+z)^{\frac{2}{a}-1}\right]\right)^{\frac{3}{2}}} \tag{21}$$

$$KU = \frac{\frac{b^{4-a}\Gamma\left(-\frac{4}{a}+1\right)}{a}\left[-\mathrm{F}(l+pc+v+2c+j)^{\frac{4}{a}-1}+\mathrm{M}(l+pc+v+2c+z)^{\frac{4}{a}-1}\right]}{\left(\frac{b^{2-a}\Gamma\left(-\frac{2}{a}+1\right)}{a}\left[-\mathrm{F}(l+pc+v+2c+j)^{\frac{2}{a}-1}+\mathrm{M}(l+pc+v+2c+z)^{\frac{2}{a}-1}\right]\right)^{2}} - 3 \tag{22}$$

Table 2: Numerical value of $\mu_1, \mu_2, \mu_3, \mu_4, \sigma^2, S$, and K of the HWFr distribution

k	c	b	a	μ_1	μ_2	μ_3	μ_4	σ^2	SK	KU
1.5	1.3	0.7	1.1	1.661728	4.798106	39.01102	4956.31	2.036766	3.711783	215.2875
		0.3	1.2	0.650964	0.661066	1.425039	17.86265	0.237312	2.651305	40.87488
	2.5	0.7	1.3	1.456806	2.324002	4.039993	7.619372	0.201718	1.140318	1.410737
		0.7	1.4	1.378262	2.054942	3.301312	5.694123	0.155336	1.120694	1.348428
1.3	1.3	0.8	1.5	1.56444	3.371826	12.10529	123.7216	0.924353	1.955138	10.88216
		0.8	1.6	1.487949	2.91341	8.643964	56.02661	0.699418	1.738243	6.600717
	2.5	0.9	1.7	1.614913	2.763094	4.99754	9.534654	0.15515	1.088085	1.24886
		0.9	1.8	1.560901	2.565588	4.432025	8.032178	0.129176	1.078504	1.22028

The moment generating function (mgf) of HWFr distribution can be founded by using the equation [20],[21]:

$$M_x(y)_{HWFr} = E(e^{yx}) = \int_{-\infty}^{\infty} e^{yx} f_{HWFr}(x,k,c,a,b)dx$$

Used series expansion for e^{yx}

$$M_x(y)_{HWFr} = \sum_{n=0}^{\infty} \frac{y^n}{n!} E(x^n) = \sum_{n=0}^{\infty} \frac{y^n}{n!} [\mu_n]$$

We but the Eq(19) in above equation we get:

$$M_x(y)_{HWFr} = \sum_{n=0}^{\infty} \frac{y^n b^{n-a} \Gamma\left(-\frac{n}{a}+1\right)}{n!\, a} \left[-\mathrm{F}(l+pc+v+2c+j)^{\frac{n}{a}-1} + \mathrm{M}(l+pc+v+2c+z)^{\frac{n}{a}-1}\right] \qquad (23)$$

3.5 Incomplete Moments

The incomplete moments of a random variable X is given by the formula [25]:

$$\mu_r(y) = \int_0^y x^r f(x) dx$$

Substituting $f_{HWFr}(x)$ for the HWFr distribution From Eq(15) into the previous equation we get:

$$\mu_r(y) = \int_0^y x^{r-(a+1)} \left[\mathrm{F} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} - \mathrm{M} e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}}\right] dx$$

$$\mu_r(y) = \mathrm{F} \int_0^y x^{r-(a+1)} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} dx - \mathrm{M} \int_0^y x^{r-(a+1)} e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}} dx$$

Let

$$u = (l+pc+v+2c+j)\frac{b^a}{x^a} \Longrightarrow x = \frac{b(l+pc+v+2c+j)^{\frac{1}{a}}}{u^{\frac{1}{a}}}$$

If $x = 0 \Longrightarrow u = 0$, if $x = y \Longrightarrow u = (l+pc+v+2c+j)\frac{b^a}{y^a} \Longrightarrow dy = -\frac{b}{a}(l+pc+v+2c+j)^{\frac{1}{a}} u^{-\frac{(1+a)}{a}} du$

$$\mathrm{F}\int_0^y x^{r-(a+1)} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} dx = \mathrm{F}\int_0^{(l+pc+v+2c+j)\frac{b^a}{x^a}} \left[\frac{b(l+pc+v+2c+j)^{\frac{1}{a}}}{u^{\frac{1}{a}}}\right]^{r-(a+1)} e^{-u} - \frac{b}{a}(l+pc+v+2c+j)^{\frac{1}{a}} u^{-\frac{(1+a)}{a}} du$$

$$\mathrm{F}\int_0^y x^{r-(a+1)} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} dx = \frac{-\mathrm{F} b^{r-a}(l+pc+v+2c+j)^{\frac{r}{a}-1}}{a} \int_0^{(l+pc+v+2c+j)\frac{b^a}{x^a}} u^{-\frac{r}{a}} e^{-u}\, du$$

$$\mathrm{F}\int_0^y x^{r-(a+1)} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} dx = \frac{-\mathrm{F} b^{r-a}(l+pc+v+2c+j)^{\frac{r}{a}-1}}{a} \Gamma\left(r+1, (l+pc+v+2c+j)\frac{b^a}{x^a}\right)$$

By same way we can find

$$\mathrm{M}\int_0^y x^{r-(a+1)}e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}}dx =$$

$$\frac{-\mathrm{M}b^{r-a}(l+pc+v+2c+z)^{\frac{r}{a}-1}}{a}\Gamma\left(r+1,(l+pc+v+2c+z)\frac{b^a}{x^a}\right)$$

$$\mu_r(y) = \frac{-\mathrm{F}b^{r-a}\mathrm{B}^{\frac{r}{a}-1}}{a}\Gamma\left(r+1,\mathrm{B}\frac{b^a}{x^a}\right) + \frac{-\mathrm{M}b^{r-a}\mathrm{A}^{\frac{r}{a}-1}}{a}\Gamma\left(r+1,\mathrm{A}\frac{b^a}{x^a}\right) \tag{24}$$

Where $\mathrm{B} = (l+pc+v+2c+j)^{\frac{r}{a}-1}$, and $\mathrm{A} = (l+pc+v+2c+z)^{\frac{r}{a}-1}$

3.6 Probability Weighted Moments

We can get the probabilistic weighted moments of HWFr distribution by using the following equation [21]:

$$\tau_{h,s} = E\left(x^h F^s(X)\right) = \int_{-\infty}^{\infty} x^h f(x) F^s(x)dx$$

By substituting $F^s_{HWFr}(x,c,k,a,b)$ and $f_{HWFr}(x,c,k,a,b)$ for the HWFr distribution From Eq(14) and Eq(15) :

$$\tau_{h,s} = \int_0^{\infty} x^{h-a-1}\left[\mathrm{KF}e^{-(2l+pc+v+2c+ic+j)\frac{b^a}{x^a}} - \mathrm{KM}e^{-(2l+pc+v+2c+ic+z)\frac{b^a}{x^a}}\right]dx$$

$$\tau_{h,s} = \mathrm{KF}\int_0^{\infty} x^{h-a-1}e^{-(2l+pc+v+2c+ic+j)\frac{b^a}{x^a}}dx - \mathrm{KM}\int_0^{\infty} x^{h-a-1}e^{-(2l+pc+v+2c+ic+z)\frac{b^a}{x^a}}dx$$

Let $y = (2l+pc+v+2c+ic+j)\frac{b^a}{x^a} \Rightarrow x = \frac{b(2l+pc+v+2c+ic+j)^{\frac{1}{a}}}{y^{\frac{1}{a}}}$

$$\Rightarrow dx = -\frac{b}{a}(2l+pc+v+2c+ic+j)^{\frac{1}{a}}y^{-\frac{(1+a)}{a}}dy$$

$$\mathrm{KF}\int_0^{\infty} x^{h-a-1}e^{-(2l+pc+v+2c+ic+j)\frac{b^a}{x^a}}dx =$$

$$-\mathrm{KF}\int_0^{\infty}\left[\frac{b(2l+pc+v+2c+ic+j)^{\frac{1}{a}}}{y^{\frac{1}{a}}}\right]^{h-a-1} e^{-y}\frac{b}{a}(2l+pc+v+2c+ic+j)^{\frac{1}{a}}y^{-\frac{(1+a)}{a}}dy$$

$$\mathrm{KF}\int_0^{\infty} x^{h-a-1}e^{-(2l+pc+v+2c+ic+j)\frac{b^a}{x^a}}dx =$$

$$-\frac{\mathrm{KF}b^{h-a}(2l+pc+v+2c+ic+j)^{\frac{h}{a}-1}}{a}\int_0^{\infty} y^{-\frac{h}{a}}e^{-y}dy$$

$$\mathrm{KF}\int_0^{\infty} x^{h-a-1}e^{-(2l+pc+v+2c+ic+j)\frac{b^a}{x^a}}dx =$$

$$-\frac{\mathrm{KF}b^{h-a}(2l+pc+v+2c+ic+j)^{\frac{h}{a}-1}}{a}\Gamma\left(-\frac{h}{a}+1\right)$$

By same way we can get:

$$\mathrm{KM}\int_0^{\infty} x^{h-a-1}e^{-(2l+pc+v+2c+ic+z)\frac{b^a}{x^a}}dx =$$

$$-\frac{\mathrm{KM}b^{h-a}(2l+pc+v+2c+ic+z)^{\frac{h}{a}-1}}{a}\Gamma\left(-\frac{h}{a}+1\right)$$

$$\tau_{h,s} = \frac{b^{h-a}\Gamma\left(-\frac{h}{a}+1\right)}{a}\left[\mathrm{KF}(2l+pc+v+2c+ic+j)^{\frac{h}{a}-1} + \mathrm{KM}(2l+pc+v+2c+ic+z)^{\frac{h}{a}-1}\right] \tag{25}$$

3.7 Disparity Measures

They are statistical instruments used to determine the extent of the disparity within a specific distribution. They offer a means to assess and contrast the level of the disparity among various datasets.

1. **Lorenz Curve**

The Lorenz curve of a random variable x is defined as [26]:

$$L_F(y) = \frac{1}{\mu}\int_{-\infty}^{y} xf(x)dx$$

We use Eq(15) in the above equation to get the form:

$$L_F(y) = \frac{1}{\mu}\int_{-\infty}^{y} x^{-a}\left[\mathrm{F}e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} - \mathrm{M}e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}}\right]dx$$

$$L_F(y) = \frac{1}{\mu}\left[\mathrm{F}\int_{-\infty}^{y} x^{-a} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} dx - \mathrm{M}\int_{-\infty}^{y} x^{-a} e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}} dx\right]$$

Let $t = (l + pc + v + 2c + j)\frac{b^a}{x^a}$, if $x = 0 \Longrightarrow t = 0$, if $x = y \Longrightarrow t = (l + pc + v + 2c + j)\frac{b^a}{y^a} \Longrightarrow dy = -\frac{b}{a}(l + pc + v + 2c + j)^{\frac{1}{a}} t^{-\frac{(1+a)}{a}}\, dt$

$\int_{-\infty}^{y} x^{-a} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} dx = \int_{o}^{(l+pc+v+2c+j)\frac{b^a}{x^a}} \left[\frac{b(2l+pc+v+2c+ic+j)^{\frac{1}{a}}}{t^{\frac{1}{a}}}\right]^{-a} e^{-t} - \frac{b}{a}(l + pc + v + 2c + j)^{\frac{1}{a}} t^{-\frac{(1+a)}{a}}\, dt$

$\int_{-\infty}^{y} x^{-a} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} dx = -\frac{b^{1-a}}{a}(l + pc + v + 2c + j)^{\frac{1}{a}-1} \int_{o}^{(l+pc+v+2c+j)\frac{b^a}{x^a}} t^{-\frac{1}{a}-2} e^{-t}\, dt$

$\int_{-\infty}^{y} x^{-a} e^{-(l+pc+v+2c+j)\frac{b^a}{x^a}} dx = -\frac{b^{1-a}}{a}(l + pc + v + 2c + j)^{\frac{1}{a}-1} \Gamma\left(-\frac{1}{a} - 1, (l + pc + v + 2c + j)\frac{b^a}{x^a}\right)$

By same way find

$\int_{-\infty}^{y} x^{-a} e^{-(l+pc+v+2c+z)\frac{b^a}{x^a}} dx = -\frac{b^{1-a}}{a}(l + pc + v + 2c + z)^{\frac{1}{a}-1} \Gamma\left(-\frac{1}{a} - 1, (l + pc + v + 2c + z)\frac{b^a}{x^a}\right)$

$$L_F(y) = \frac{1}{\mu a}\left[-\mathrm{F}b^{1-a}\mathrm{B}^{\frac{1}{a}-1}\Gamma\left(-\frac{1}{a} - 1, \mathrm{B}\frac{b^a}{x^a}\right) + \mathrm{M}b^{1-a}\mathrm{A}^{\frac{1}{a}-1}\Gamma\left(-\frac{1}{a} - 1, \mathrm{A}\frac{b^a}{x^a}\right)\right] \tag{26}$$

Where $\mathrm{B} = (l + pc + v + 2c + j)^{\frac{r}{a}-1}$, and $\mathrm{A} = (l + pc + v + 2c + z)^{\frac{r}{a}-1}$

2. Bonferroni Curve

The Bonferroni curve of a random variable x is defined as [22]:

$$\mathcal{B}_F(y) = \frac{L_F(y)}{F_{HWFr}(x, c, k, a, b)}$$

By substation Eq(7) and Eq(26) in above equation we get:

$$\mathcal{B}_F(y) = \frac{\left[-\mathrm{F}b^{1-a}\mathrm{B}^{\frac{1}{a}-1}\Gamma\left(-\frac{1}{a} - 1, \mathrm{B}\frac{b^a}{x^a}\right) + \mathrm{M}b^{1-a}\mathrm{A}^{\frac{1}{a}-1}\Gamma\left(-\frac{1}{a} - 1, \mathrm{A}\frac{b^a}{x^a}\right)\right]}{\mu a\left(1 - \mathrm{E}e^{-\frac{b^a}{x^a}(l+pc)}\right)} \tag{27}$$

3.8 Entropy

Entropy refers to the measurement of disorder or randomness in a system. It is a crucial metric for measuring uncertainty. Many forms of entropy are valuable for assessing risk and doing reliability analysis.

Entropy is commonly used concept in various fields, including engineering, econometrics, and financial mathematics. It works as a method to measure the degree of variance or uncertainty linked to a random variable. The entropy of a random variable x, as determined by its density function $f(x)$, determine the level of uncertainty inherent in the data. Using entropy has gained importance in various scientific and technical domains.

A. Rényi Entropy

The Rényi's Entropy of a random variable x is defined as [20]:

$$I_R(\delta)_{HWFr} = \frac{1}{1-\delta} log \int_{0}^{\infty} f(x, k, c, a, b)^{\delta} dx$$

By substitution Eq(16) in above equation we get:

$$I_R(\delta)_{HWFr} = \frac{1}{1-\delta} log \int_0^\infty x^{-\delta(a+1)} \sum_{m=0}^{z} (-1)^m \binom{z}{m} \mathrm{F.M}\, e^{-(\delta l+\delta pc+\delta v+\delta 2c+\delta z-mz)\frac{b^a}{x^a}} dx$$

$$I_R(\delta)_{HWFr} = \frac{1}{1-\delta} \sum_{m=0}^{z} (-1)^m \binom{z}{m} \mathrm{F.M}\, log \int_0^\infty x^{-\delta(a+1)} e^{-(\delta l+\delta pc+\delta v+\delta 2c+\delta z-mz)\frac{b^a}{x^a}} dx$$

Let $y = (\delta l + \delta pc + \delta v + \delta 2c + \delta z - mz)\frac{b^a}{x^a} \Longrightarrow x^a = \frac{b^a(l+pc+v+2c+z)}{y} \Longrightarrow x = \frac{b(\delta l+\delta pc+\delta v+\delta 2c+\delta z-mz)^{\frac{1}{a}}}{y^{\frac{1}{a}}} \Longrightarrow dx =$

$$-\frac{b}{a}(\delta l + \delta pc + \delta v + \delta 2c + \delta z - mz)^{\frac{1}{a}} y^{-\frac{(1+a)}{a}}\, dy$$

$$I_R(\delta)_{HWFr} = \frac{1}{(1-\delta)} \sum_{m=0}^{z} (-1)^m \binom{z}{m} \mathrm{F.M} \log \left[\frac{-(\delta l + \delta pc + \delta v + \delta 2c + \delta z - mz)^{-\delta\left(1+\frac{1}{a}\right)+\frac{1}{a}}}{ab^{\delta(a+1)-1}} \int_0^\infty y^{\delta\left(1+\frac{1}{a}\right)-\frac{1}{a}-1} e^{-y} dy \right]$$

$$I_R(\delta)_{HWFr} = \frac{1}{(1-\delta)} \sum_{m=0}^{z} (-1)^m \binom{z}{m} \mathrm{F.M} \times \log \left[\frac{-(\delta l + \delta pc + \delta v + \delta 2c + \delta z - mz)^{-\delta\left(1+\frac{1}{a}\right)+\frac{1}{a}}}{ab^{\delta(a+1)-1}} \Gamma\left(\delta + \frac{\delta-1}{a}\right) \right] \quad (28)$$

B. Arimoto Entropy

Arimoto entropy is a quantitative measurement of the amount of information or unpredictability in a given distribution. The AEN measure is calculated using the following equation [27]:

$$A(\delta)_{HWFr} = \frac{\delta}{1-\delta}\left(\left[\int_0^\infty f(x,k,c,a,b)^\delta dx\right]^{\frac{1}{\delta}} - 1\right)$$

Form Eq(28) we get:

$$\int_0^\infty f(x,k,c,a,b)^\delta dx = \frac{-(\delta l + \delta pc + \delta v + \delta 2c + \delta z - mz)^{-\delta\left(1+\frac{1}{a}\right)+\frac{1}{a}}}{ab^{\delta(a+1)-1}} \Gamma\left(\delta + \frac{\delta-1}{a}\right)$$

$$A(\delta)_{HWFr} = \frac{\delta}{1-\delta}\left(\left[\frac{-(\delta l+\delta pc+\delta v+\delta 2c+\delta z-mz)^{-\delta\left(1+\frac{1}{a}\right)+\frac{1}{a}}}{ab^{\delta(a+1)-1}} \Gamma\left(\delta + \frac{\delta-1}{a}\right)\right]^{\frac{1}{\delta}} - 1\right) \quad (29)$$

Form Eq(29) we get:

C. Havrda and Charvat Entropy

$$HC(\delta)_{HWFr} = \frac{1}{2^{1-\delta}-1}\left(\left[\int_0^\infty f(x,k,c,a,b)^\delta dx\right]^{\frac{1}{\delta}} - 1\right)$$

$$HC(\delta)_{HWFr} = \frac{1}{2^{1-\delta}-1}\left(\left[\frac{-(\delta l+\delta pc+\delta v+\delta 2c+\delta z-mz)^{-\delta\left(1+\frac{1}{a}\right)+\frac{1}{a}}}{ab^{\delta(a+1)-1}} \Gamma\left(\delta + \frac{\delta-1}{a}\right)\right]^{\frac{1}{\delta}} - 1\right) \quad (30)$$

D. Tsallis Entropy

$$T(\delta)_{HWFr} = \frac{1}{\delta-1}\left(1 - \int_0^\infty f(x,k,c,a,b)^\delta dx\right)$$

Form Eq(28) we get:

$$T(\delta)_{HWF} = \frac{1}{\delta - 1}\left(1 + \frac{(\delta l + \delta pc + \delta v + \delta 2c + \delta z - mz)^{-\delta\left(1+\frac{1}{a}\right)+\frac{1}{a}}}{ab^{\delta(a+1)-1}}\Gamma\left(\delta + \frac{\delta - 1}{a}\right)\right) \quad (32)$$

E. Estimation

The parameters of the HWFr distribution are estimated using the maximum likelihood estimation approach. We derive the log-likelihood function for a random sample $x_1, x_2, \ldots, x_n$. The distribution follows the pdf of the HWFr distribution [28].

$$L(\theta, x) = \prod_{i=1}^{n} f_{HWFr}(x_i, k, c, a, b)$$

$$L(\theta, x_i) = \prod_{i=1}^{n} kc\, ab^a x^{-(a+1)} e^{-\left(\frac{b^a}{x^a}\right)}\left[\frac{e^{-\left(\frac{b^a}{x^a}\right)}}{1 - e^{-\left(\frac{b^a}{x^a}\right)}} - log\left(1 - e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]$$

$$* \left[-e^{-\left(\frac{b^a}{x^a}\right)}.\,log\left(1 - e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^{c-1} e^{\left(-k\left[-e^{-\left(\frac{b^a}{x^a}\right)}.log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right)\right]^{c}\right)}$$

The log-likelihood function L is obtained as:

$$\begin{aligned}L = nlog(k) + nlog(c) + nlog(a) + anlog(b) - (a+1)\textstyle\sum_{i=1}^{n} \frac{b^a}{x_i^a} + \sum_{i=1}^{n} \log\left[\frac{e^{-\frac{b^a}{x_i^a}}}{1-e^{-\frac{b^a}{x_i^a}}} -\right.\\ \left. log\left(1 - e^{-\frac{b^a}{x_i^a}}\right)\right] + (c-1)\textstyle\sum_{i=1}^{n} \log\left(-e^{-\frac{b^a}{x_i^a}}.\,log\left(1 - e^{-\frac{b^a}{x_i^a}}\right)\right) - k\sum_{i=1}^{n}\left[-e^{-\frac{b^a}{x_i^a}}.\,log\left(1 -\right.\right.\\ \left.\left. e^{-\frac{b^a}{x_i^a}}\right)\right]^{c}\end{aligned} \quad (31)$$

By partially deriving the above equation with respect to the distribution parameters

$$\frac{\partial L}{\partial k} = \frac{n}{k} - \sum\nolimits_{i=1}^{n}\left[-e^{-\frac{b^a}{x_i^a}}.\,log\left(1 - e^{-\frac{b^a}{x_i^a}}\right)\right]^{c} \quad (32)$$

$$\begin{aligned}\frac{\partial L}{\partial c} = &\\ \frac{n}{c} + \textstyle\sum_{i=1}^{n} \log\left(-e^{-\frac{b^a}{x_i^a}}.\,log\left(1 - e^{-\frac{b^a}{x_i^a}}\right)\right) - k\sum_{i=1}^{n}\left[-e^{-\frac{b^a}{x_i^a}}.\,log\left(1 - e^{-\frac{b^a}{x_i^a}}\right)\right]^{c} \ln\left[-e^{-\frac{b^a}{x_i^a}}.\,log\left(1 -\right.\right.\\ \left.\left. e^{-\frac{b^a}{x_i^a}}\right)\right]\end{aligned} \quad (33)$$

$$\begin{aligned}\frac{\partial L}{\partial a} = &\\ \frac{n}{a} + nlog(b) - \textstyle\sum_{i=1}^{n}(a+1)b^a x_i^{-a}\ln b + \sum_{i=1}^{n}(a+1)b^a x_i^{-a}\ln x_i - \sum_{i=1}^{n}\frac{1}{e^{-\frac{b^a}{x_i^a}}.log\left(1-e^{-\frac{b^a}{x_i^a}}\right)}\left[\left(c -\right.\right.\end{aligned} \quad (34)$$

$$1)\Bigg(-\Bigg((-b^a x_i^{-a}\ln b+b^a x_i^{-a}\ln x_i)e^{-\frac{b^a}{x_i^a}}\Bigg)log\left(1-e^{-\frac{b^a}{x_i^a}}\right)+$$

$$e^{-\frac{b^a}{x_i^a}}\left(\frac{(-b^a x_i^{-a}\ln b+b^a x_i^{-a}\ln x_i)e^{-\frac{b^a}{x_i^a}}}{1-e^{-\frac{b^a}{x_i^a}}}\right)\Bigg)\Bigg]+$$

$$\sum_{i=1}^{n}\frac{1}{e^{-\frac{b^a}{x_i^a}}.log\left(1-e^{-\frac{b^a}{x_i^a}}\right)}\Bigg[kc\left[-e^{-\frac{b^a}{x_i^a}}.log\left(1-e^{-\frac{b^a}{x_i^a}}\right)\right]^c\Bigg(-\Bigg((-b^a x_i^{-a}\ln b+$$

$$b^a x_i^{-a}\ln x_i)e^{-\frac{b^a}{x_i^a}}\Bigg)log\left(1-e^{-\frac{b^a}{x_i^a}}\right)+e^{-\frac{b^a}{x_i^a}}\left(\frac{(-b^a x_i^{-a}\ln b+b^a x_i^{-a}\ln x_i)e^{-\frac{b^a}{x_i^a}}}{1-e^{-\frac{b^a}{x_i^a}}}\right)\Bigg)\Bigg]$$

$$\frac{\partial L}{\partial b}=$$

$$\frac{an}{b}-(a+1)\sum_{i=1}^{n}\frac{ab^{a-1}}{x_i^a}-(c-1)\sum_{i=1}^{n}\frac{e^{-\frac{b^a}{x_i^a}}x_i^{-a}ab^{a-1}\left[\frac{e^{-\frac{b^a}{x_i^a}}}{1-e^{-\frac{b^a}{x_i^a}}}-log\left(1-e^{-\frac{b^a}{x_i^a}}\right)\right]}{\left(e^{-\frac{b^a}{x_i^a}}.log\left(1-e^{-\frac{b^a}{x_i^a}}\right)\right)}+ \tag{35}$$

$$kc\sum_{i=1}^{n}\left[-e^{-\frac{b^a}{x_i^a}}.log\left(1-e^{-\frac{b^a}{x_i^a}}\right)\right]^c\frac{\left[e^{-\frac{b^a}{x_i^a}}x_i^{-a}ab^{a-1}\,log\left(1-e^{-\frac{b^a}{x_i^a}}\right)-e^{-\frac{b^a}{x_i^a}}\left(\frac{e^{-\frac{b^a}{x_i^a}}x_i^{-a}ab^{a-1}}{1-e^{-\frac{b^a}{x_i^a}}}\right)\right]}{\left(e^{-\frac{b^a}{x_i^a}}.log\left(1-e^{-\frac{b^a}{x_i^a}}\right)\right)}$$

Solving the non-linear equations for $\frac{\partial L}{\partial \alpha}=0,\frac{\partial L}{\partial \lambda}=0,$ and $\frac{\partial L}{\partial \gamma}=0$ gives the ML estimates of parameters $\alpha,\beta,$ and λ respectively. The solution could not be found by analysis. It could only be found by numerical methods. These methods use software like R, MAPLE, SAS, and so on.

F. Simulation

Simulation is crucial in signal processing, extreme analysis, and air temperature adjustment. Signal processing uses simulation to understand ultrasound data. Extreme analysis, especially meteorological analyses, uses simulation to evaluate extreme estimates From diverse methodologies and data sets. Simulation helps researchers understand their domains' processes and features.

The performance of maximum likelihood estimators (MLEs) for the HWFr distribution is evaluated through a Monte Carlo simulation study using the R package. The sample sizes considered in the study are n = 50, 100, and 150. We generate N = 1000 samples for the true parameter values listed in Table 3. The resulting MLEs for the model parameters are averaged to obtain the mean values, and the corresponding bias and root mean squared errors (RMSEs) are calculated. The bias and RMSE for a specific estimated parameter, denoted as $\hat{\gamma}$, are given by:

$$bias(\hat{\gamma})=\frac{\sum_{i=1}^{N}\hat{\gamma}_i}{N}-\gamma,\;\; RMSE(\hat{\gamma})=\sqrt{\frac{\sum_{i=1}^{N}(\hat{\gamma}_i-\gamma)^2}{N}}.$$

The consistency of all estimators is shown in Tables 3. As the sample size rises, the average parameter estimations converge towards the real parameter values. Furthermore, the mean square errors (MSEs) exhibit a reduction in magnitude as the sample size grows.

Table.3 outcomes of Monte Carlo simulations conducted for the HWFr distribution

		$(k = 0.9, c = 0.8, a = 1.1, b = 1.3)$			$(k = 0.7, c = 0.5, a = 0.6, b = 0.8)$		
parameter	Sample Size	Mean	RMSE	bias	Mean	RMSE	bias
k	50	0.60714	0.47951	0.29285	0.58857	0.4644665	0.1114224
	100	0.67595	0.53024	0.22404	0.60008	0.40073458	0.0999193
	150	0.74503	0.63774	0.15496	0.63818	0.50762186	0.0618101
c	50	0.34401	0.52795	0.45598	0.33991	0.2768812	0.1600838
	100	0.39099	0.48609	0.40900	0.37414	0.28461204	0.1258505
	150	0.45292	0.54761	0.34707	0.41679	0.27610623	0.0832081
a	50	1.70184	0.81451	0.60184	0.79486	0.3410188	0.1948696
	100	1.63957	0.75850	0.53957	0.77444	0.3041937	0.1744471
	150	1.60084	0.77071	0.50084	0.75570	0.30339292	0.1557019
b	50	1.25407	0.25562	0.04592	0.77792	0.1889598	0.0220720
	100	1.28292	0.36158	0.01707	0.78832	0.21981466	0.0116768
	150	1.28662	0.29772	0.01337	0.79734	0.1779633	0.0026533
		$(k = 1.4, c = 1.6, a = 1.9, b = 1.4)$			$(k = 2.2, c = 2.5, a = 2.3, b = 2.8)$		
parameter	Sample Size	Mean	RMSE	bias	Mean	RMSE	bias
k	50	1.16053	0.75538	0.23946	1.16168	1.38541	1.038319
	100	1.16616	0.67009	0.23383	1.17747	1.284248	1.022526
	150	1.36002	1.43215	0.03997	1.36623	1.9381993	0.8337618
c	50	0.66778	0.98079	0.93221	0.89638	1.810555	1.6036158
	100	0.68082	0.95267	0.91917	1.09565	1.818058	1.404345
	150	0.72448	1.11844	0.87551	1.12288	1.844467	1.377115
a	50	3.28018	1.84631	1.38018	4.84349	3.428097	2.543498
	100	3.03768	1.38178	1.13768	4.41473	2.928998	2.114734
	150	3.20670	1.65285	1.3067	4.27030	3.160557	1.970305
b	50	1.51742	0.54694	0.11742	2.44718	1.073885	0.3528194
	100	1.50343	0.43736	0.10343	2.61179	0.6859866	0.1882019
	150	1.46297	0.40087	0.06297	2.63100	0.850409	0.1689939

From table 3, we note that means converge to true values of parameters as sample as the sample size (n) increases, indicating the consistency of the estimations. For example:

- For the first set $(k = 0.9)$ the means improve from 0.60714 $(n = 50)$ to 0.74503 $(n = 150)$.
- For third set $(a = 1.9)$, the means improve from 3.28018 $(n = 50)$ to 3.20670 $(n = 150)$.

We note that skewness decreases with increasing sample size, supporting the lack of asymptotic bias. Also note that RMSE generally decreases with increasing sample size, confirming the efficiency of estimators. Some estimators (such as c in first set) require larger samples to converge better, as skewness remains relatively high even at n=150. Meanwhile, the estimators for the b parameter are typically more accurate and less skewed than others, as seen across all sets. The results confirm that the MLE estimators for the HWF distribution exhibit good properties such as consistency and asymptotic unbiasedness, especially with increasing sample size. This supports the use of this method for estimating distribution parameters in practical applications. However, some parameters (such as c and a) may require larger samples to achieve higher accuracy.

G. **Application**

We demonstrate a pragmatic implementation using two sets of data. The efficacy of the HWFr distribution in accurately fitting data is shown. The application showcases the benefits of HWFr and its excellent compatibility with the data. Table 4 presents a comparison between HWFr and other distributions. This process is performed on the used data.

This comparison employs eight metrics. The two statistical measures used are the Kolmogorov-Smirnov statistic (KS) and the Anderson-Darling statistic (A). The Cramér-von Mises statistic (W) and the HQIC, BIC, AIC, and CAIC information criteria are also considered. Additionally, it utilises the p-value obtained From the Kolmogorov-Smirnov test. These metrics are often used to assess the quality of fit.

Tables 5 and 6 show the HWFr distribution that has the lowest values for AIC, CAIC, and BIC. They are compared to the values of non-overlapping distributions. Furthermore, statistical tests such as the Anderson-Darling, Watson, and Kolmogorov-Smirnov tests demonstrate that the distribution of the HWFr accurately matches the patterns seen in both data I and data II.

Table.4 Comparative distributions

Distribution	**CDF**
Beta Fréchet distribution [20]	$\beta\left(e^{-\left(\frac{b^a}{x^a}\right)}, k, c\right)$
Kumaraswamy Fréchet distribution (New)	$1-\left(1-\left(e^{-\left(\frac{b^a}{x^a}\right)}\right)^k\right)^c$
Exponential Generalized Exponential Fréchet distribution (New)	$\left(1-\left(e^{-\left(\frac{b^a}{x^a}\right)}\right)^k\right)^c$
Log Gamma Fréchet	$1-\Gamma\left(-c\log\left(1-e^{-\left(\frac{b^a}{x^a}\right)}\right), k\right)$
[0,1]Truncated Exponentiated Exponential Fréchet (New)	$\dfrac{\left(1-e^{-ke^{-\left(\frac{b^a}{x^a}\right)}}\right)^c}{(1-e^{-k})^c}$
Fréchet distribution [24]	$e^{-\left(\frac{b^a}{x^a}\right)}$

- **The First Dataset I**

The dataset used in this investigation comprises the longevity of 72 guinea pigs. The guinea pigs were afflicted with highly infectious tubercle bacilli. Scientists quantify the duration of survival in terms of days. Bjerkedal [29] performed the first observation and documentation of this dataset.

0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58 , 5.55

Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
1	72	1.77	1.04	1.5	1.64	0.72	0.1	5.55	5.45	1.3	1.83	0.12

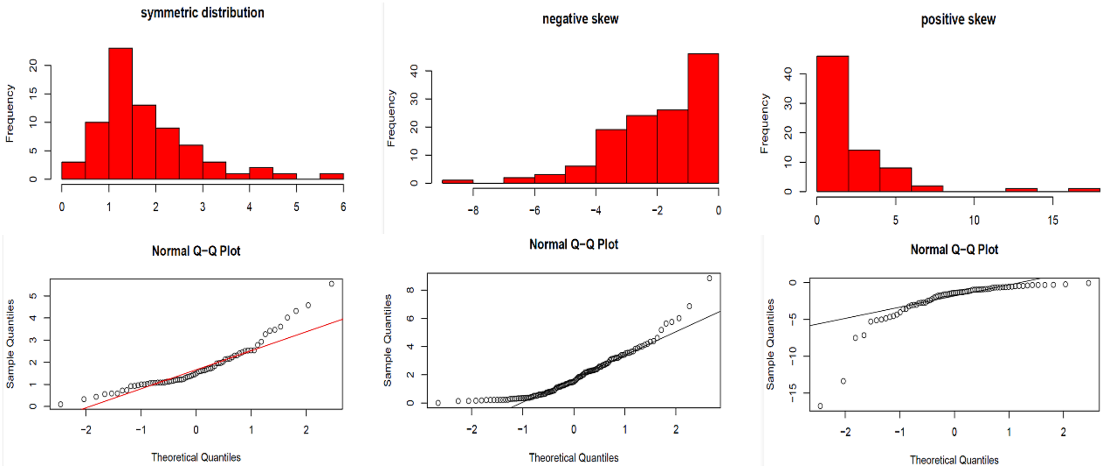


Figure.3 Data I distribution and its QQ-plot

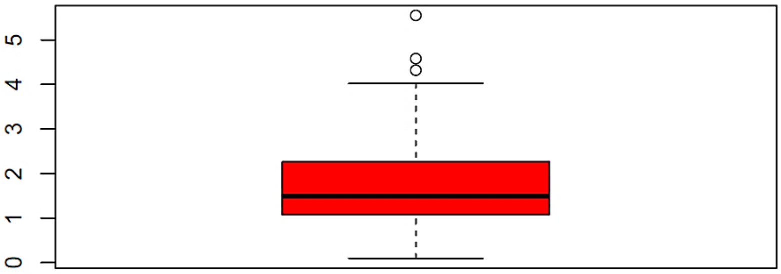


Figure.4 Box plot for Data I

Table.5 Estimates of models for data I

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
HWFr	$\hat{c}$: 1.4481920 $\hat{k}$: 3.3123383 $\hat{a}$: 0.2274464 $\hat{b}$: 0.5163086	95.64765	199.3066	199.9036	208.4133	202.932	0.09157692	0.5996763	0.1087735	0.3618078
BeFr	$\hat{c}$: 1.3683981 $\hat{k}$: 4.9211484 $\hat{a}$: 2.9049637 $\hat{b}$: 0.7243188	102.7963	213.6406	214.2376	222.7473	217.266	0.1989575	1.355844	0.1505879	0.0763555
KuFr	$\hat{c}$: 2.1050208 $\hat{k}$: 8.9649266 $\hat{a}$: 1.9957996 $\hat{b}$: 0.6450534	99.57312	207.2478	207.8448	216.3545	210.8732	0.1374487	0.9563294	0.1363578	0.137431
EGFr	$\hat{c}$: 6.7019951 $\hat{k}$: 0.9951016 $\hat{a}$: 4.7880547 $\hat{b}$: 0.7017667	100.8311	209.6678	210.2648	218.7745	213.2932	0.1596745	1.104107	0.1384955	0.1262907
LGFr	$\hat{c}$: 1.1107672 $\hat{k}$: 5.3546042 $\hat{a}$: 3.5956842 $\hat{b}$: 0.7329769	102.11	212.2587	212.8557	221.3653	215.8841	21.50031	140.4778	0.9999987	0
TEEFr	$\hat{c}$: 44.3248058 $\hat{k}$: 0.8273607 $\hat{a}$: 33.3540295 $\hat{b}$: 0.4751488	96.22994	200.4599	201.0569	209.5665	204.0853	0.09117708	0.6164712	0.1136481	0.3102217
Fr	$\hat{a}$: 1.052192 $\hat{b}$: 1.171994	118.1178	240.2356	240.4095	244.7889	242.0483	0.5284355	3.310618	0.1991305	0.0066251

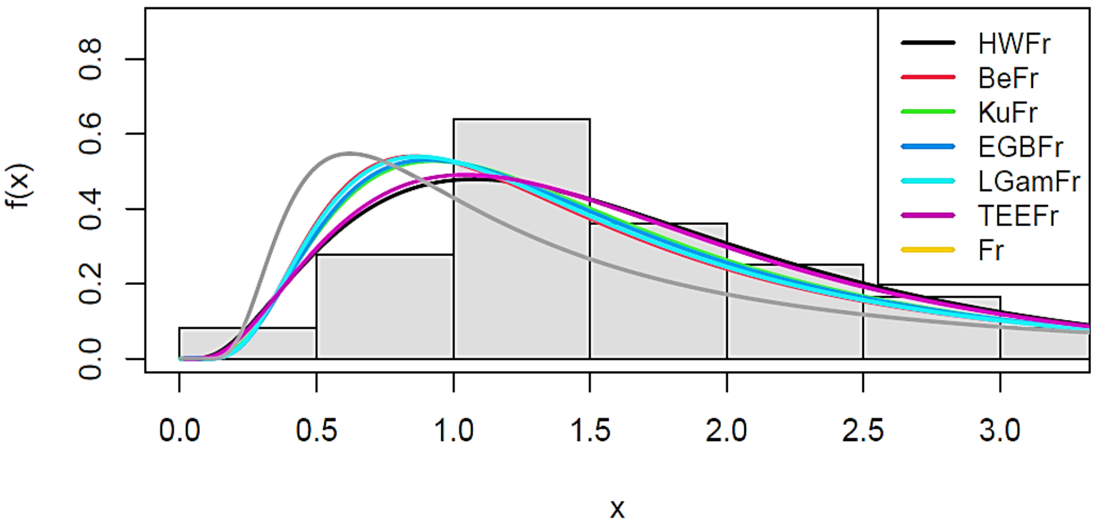
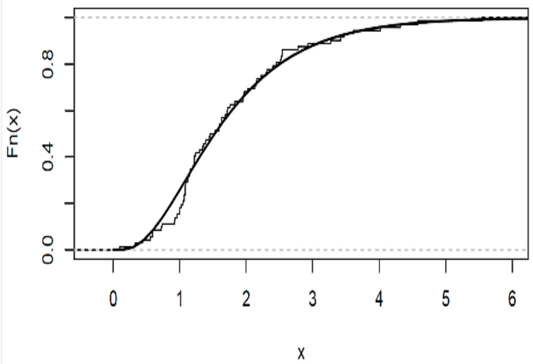
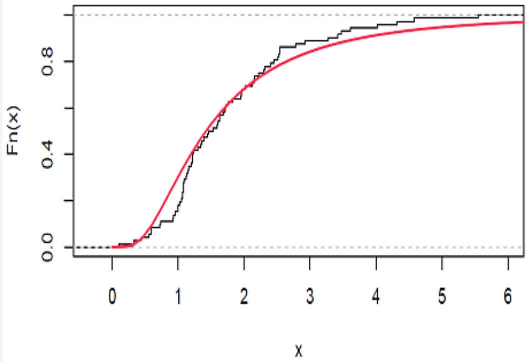


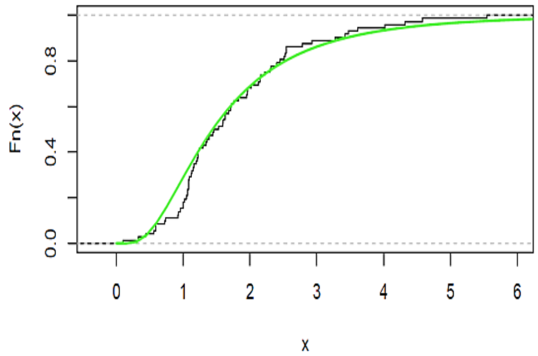
Figure.5 Fitted densities for Data I



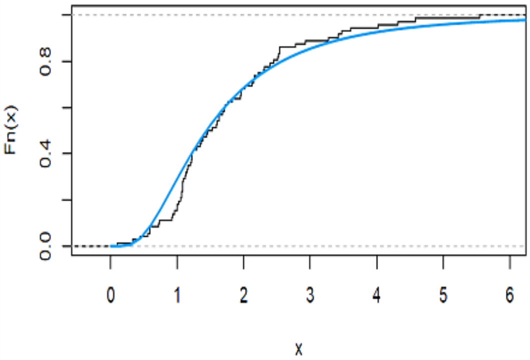
Empirical CDF for HWFr



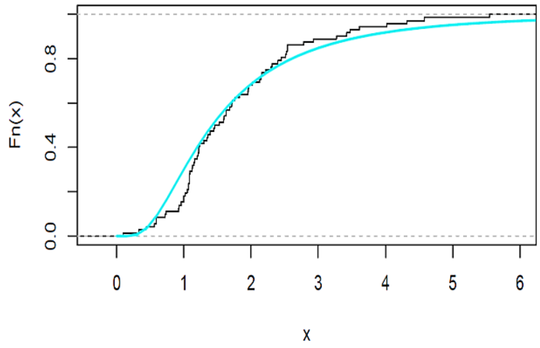
Empirical CDF for BeFr



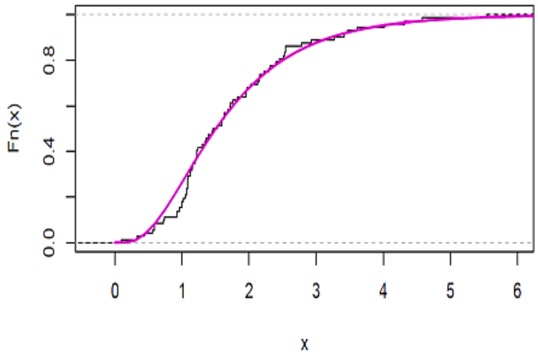
Empirical CDF for KuFr



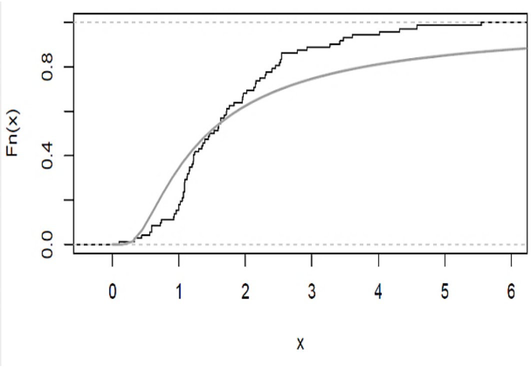
Empirical CDF for EGFr



Empirical CDF for LGFr



Empirical CDF for TEEFr



Empirical CDF for Fr

Figure.6 Empirical CDF for Data I

- **The Second Dataset II**

The remission times (in months) of 128 patients suffering From bladder cancer. Recently, the data set is used by [30],[31], the data set values were:

0.08, 0.20, 0.40, 0.050, 0.081, 0.090, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.70, 3.82, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 4.51, 4.87, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 5.41, 5.49, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 6.97, 7.09, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 7.87, 7.93, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 9.74, 10.06, 8.26, 9.74, 10.06, 8.26, 9.74, 10.06,8.26, 9.74, 12.03, 12.07, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05

Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
1	128	9.45	10.53	6.54	7.52	5.62	0.05	79.05	79	3.23	15.06	0.93

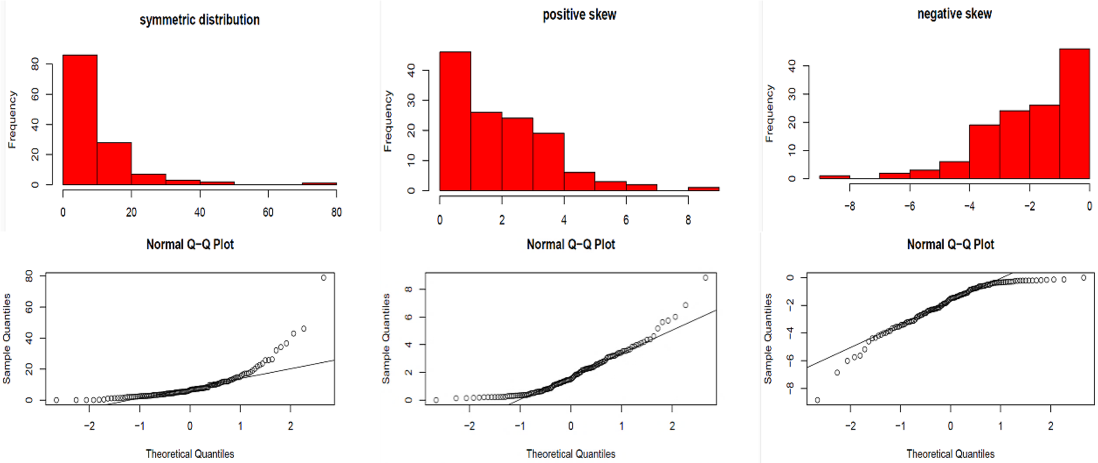


Figure.7 Data II distribution and its QQ-plot

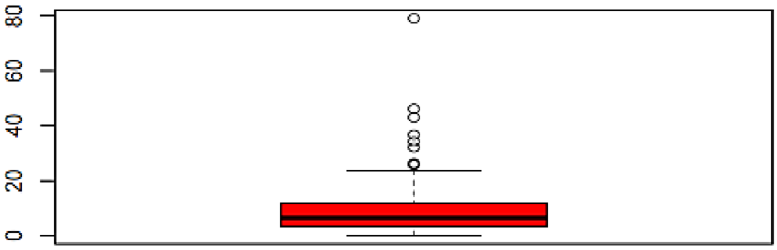


Figure.8 Box plot for Data II

Table.6 Estimates of models for data II

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
HWFr	$\hat{c}$: 1.02777627 $\hat{k}$: 4.07786551 $\hat{a}$: 0.07152354 $\hat{b}$: 0.24932628	414.675	837.3548	837.6827	848.7316	841.9771	0.1722693	1.382277	0.0844761	0.325052
BeFr	$\hat{c}$: 2.9258356 $\hat{k}$: 3.1969147 $\hat{a}$: 2.3598177 $\hat{b}$: 0.3989486	441.3903	890.9892	891.3171	902.366	895.6115	0.8610914	5.724754	0.162881	0.0023684
KuFR	$\hat{c}$: 2.7956281 $\hat{k}$: 4.4383600 $\hat{a}$: 2.0778270 $\hat{b}$: 0.3949976	433.7467	875.6575	875.9854	887.0342	880.2797	0.6422524	4.431174	0.1372944	0.0166609
EGFr	$\hat{c}$: 3.6230026 $\hat{k}$: 3.6990283 $\hat{a}$: 4.2698267 $\hat{b}$: 0.3282388	441.5489	891.1161	891.4439	902.4928	895.7383	0.864143	5.745746	0.1599324	0.0030161
LGFr	$\hat{c}$: 2.9530303 $\hat{k}$: 4.1986412 $\hat{a}$: 2.5209823 $\hat{b}$: 0.3862713	439.9068	887.9424	888.2703	899.3192	892.5646	34.82408	242.5317	0.9999823	0
TEEFr	$\hat{c}$: 5.911888 $\hat{k}$: 5.104056 $\hat{a}$: 5.807561 $\hat{b}$: 0.296075	456.4447	920.8923	921.2202	932.2691	925.5146	1.3077	8.254507	0.1832348	0.0003956
Fr	$\hat{a}$: 2.6581897 $\hat{b}$: 0.6154614	461.0474	926.1037	926.2005	931.7921	928.4149	1.408272	8.776213	0.2036627	5.3150e-05

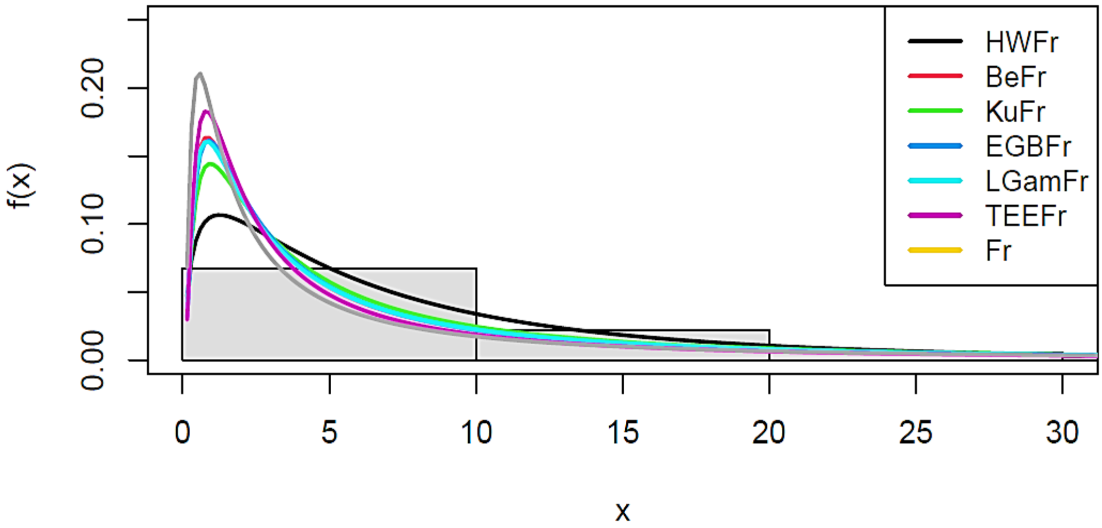
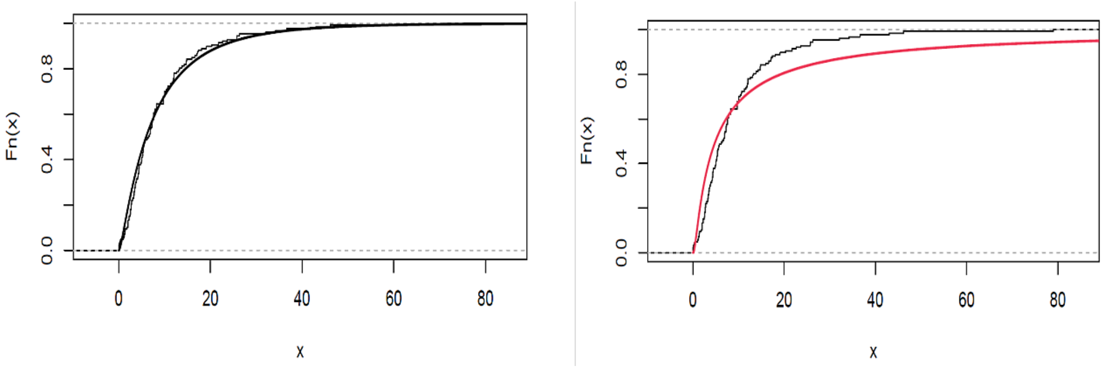
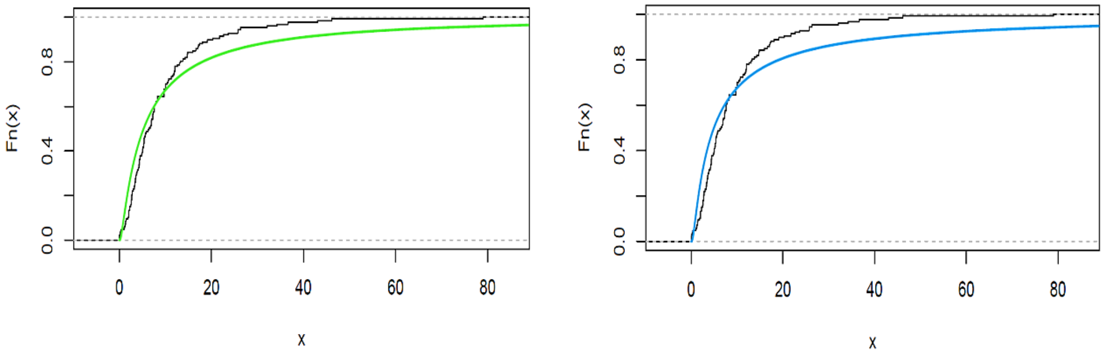


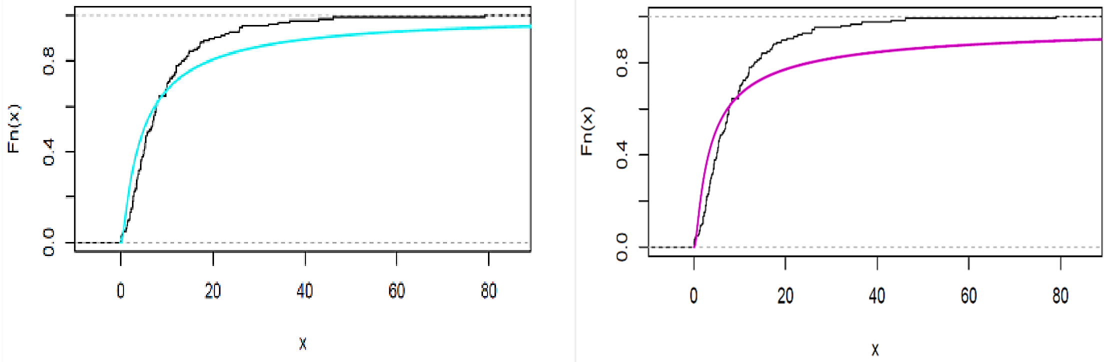
Figure.9 Fitted densities for Data I



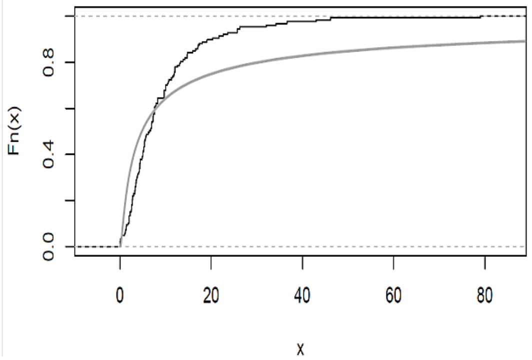
Empirical CDF for HWFr Empirical CDF for BeFr



Empirical CDF for KuFr Empirical CDF for EGFr



Empirical CDF for LGFr Empirical CDF for TEEFr



Empirical CDF for Fr

Figure.10 Empirical CDF for Data I

4. Conclusion

The research introduced a novel continuous distribution, namely the Hybrid Weibull Frechet distribution, which is derived From the Hybrid Weibull-G family. This distribution is characterised by four characteristics. The statistical properties of this distribution were shown via practical application on two kinds of actual data, showcasing the efficiency and adaptability of the novel distribution. The values presented in Tables 5 and 6, as well as Figures 5,6,9, and 10, depict the comparison between the new distribution and various other distributions, both new and established. This comparison was conducted using information standards and visual inspection. The results confirm that the HWFr distribution aligns well with the data, outperforming the other distributions.

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