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Compare Some Shrinkage Bayesian Estimators for Kumaraswamy Distribution with Simulation

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ABSTRACT

The Kumaraswamy distribution, a flexible probability distribution with wide applications in various fields, has attracted significant attention in statistical research.
This study focuses on the comparison of shrinkage Bayesian estimators for the parameters of the Kumaraswamy distribution through simulations experiments.
We explore and evaluate different shrinkage Bayesian estimation methods, aiming to enhance the accuracy and efficiency of parameter estimation for the Kumaraswamy distribution with different loss functions.
Through a comprehensive simulation study, we assess the performance of these estimators in terms of mean squared error under varying sample sizes and parameter values. Our findings provide valuable insights into the robustness and efficiency of the considered shrinkage Bayesian estimators for the Kumaraswamy distribution. The simulation results contribute to the understanding of the efficiency, aiding practitioners in choosing appropriate estimation methods for specific scenarios. Additionally, the study discusses the implications of these findings for real-world applications where the Kumaraswamy distribution is commonly employed.

1. Introduction

In the realm of statistical modelling and parameter estimation, the selection of appropriate estimation methods is crucial for achieving accurate and reliable results. Bayesian statistics provides a powerful framework for inference, offering a systematic approach to incorporate prior knowledge into the estimation process. Within this Bayesian framework, the use of shrinkage estimators has gained prominence due to their ability to strike a balance between the sample information and prior beliefs.
This research endeavours to explore and compare various shrinkage Bayesian estimators applied to the Kumaraswamy distribution, a versatile probability distribution widely employed in diverse fields. The Kumaraswamy distribution, with its flexibility in modelling a spectrum of shapes, presents an intriguing context for evaluating the performance of Bayesian estimators under different scenarios.
The motivation for this study stems from the need to enhance the accuracy and efficiency of parameter estimation for the Kumaraswamy distribution, particularly in situations where traditional estimators may exhibit shortcomings. Shrinkage estimators, characterized by their tendency to "shrink" towards a central value, offer a promising avenue for improving estimation performance

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by borrowing strength from both the observed data and prior information.

Simulations play a pivotal role in this study, allowing us to systematically explore the behavior of different shrinkage Bayesian estimators under controlled conditions. By varying sample sizes and levels of shrinkage, we seek to uncover patterns and trends that inform the selection of optimal estimators for specific scenarios. The outcomes of this research contribute not only to the understanding of Bayesian estimation in the context of the Kumaraswamy distribution but also offer practical guidance for researchers and practitioners engaged in statistical modeling with bounded variables.

In summary, this research endeavors to shed light on the performance of various shrinkage Bayesian estimators for the Kumaraswamy distribution through a rigorous simulation-based investigation. The findings are expected to contribute to the advancement of statistical methodologies and aid in the selection of robust estimation techniques in applications where the Kumaraswamy distribution is a suitable model.

In this field, a lot of research has been done

The research of (Sugasawa, S., Kubokawa, T., & Ogasawara, K.) in (2017)

The research stated that prior distribution may inflate estimation error and therefore it is recommended to consider an uncertain prior distribution which is expressed as a combination of a one-point distribution and a proper prior distribution. The research involved the development of an empirical Bayesian approach to estimating area-level means, using the uncertainty prior distribution in the context of a natural exponential family, which is called the uncertainty Bayes (EUB) method. The regression model considered in this paper includes Poisson-gamma, beta binomial, and Phi-Heriot model, which are commonly used in small area estimation. Hyperparameter estimators based on marginal likelihood are obtained using the well-known expectation maximization algorithm, and EUB estimators for area means are proposed. To evaluate the estimated EUB risk, we derive an unbiased second-order conditional mean square error estimator using some numerical calculation techniques. Through simulation studies and in real data applications, the performance of the EUB estimator was evaluated and compared with the usual empirical Bayes estimator [13].

The research of (Sousa, A. R. d. S.)in (2023)

This work proposes a wavelet deflation rule under the asymmetric LINEX loss function and a mixture of point mass function at zero and logistic distribution as prior distribution of wavelet coefficients in nonparametric regression model with Gaussian error. Underestimating the wavelet coefficient can lead to poor detection of unknown function features such as peaks, discontinuities, and oscillations. It can also occur under asymmetrically distributed wavelet coefficients. Thus the proposed rule is appropriate when overestimation and underestimation have asymmetric losses. Statistical properties norm such as squared bias, variance, frequents hazard, and Bayesian hazard are obtained. Simulation studies are performed to evaluate the performance of the rule against standard methods and application A real data set including infrared spectra is provided [14].

The research of (Sewell, D. K.)in (2024)

The research stated that It is common to hold prior beliefs that are not characterized by points in the parameter space but instead are relational in nature and can be described by a linear subspace. While some previous work has been done to account for such prior beliefs, the focus has primarily been on point estimators within a regression framework. We argue, however, that prior beliefs about parameters ought to be encoded into the prior distribution rather than in the formation of a point estimator. In this way, the prior beliefs help shape all inference. Through exponential tilting, we propose a fully generalizable method of taking existing prior information from, e.g., a pilot study, and combining it with additional prior beliefs represented by parameters lying on a linear subspace. We provide computationally efficient algorithms for posterior inference that, once inference is made using a non-tilted prior, does not depend on the sample size .approach on an antihypertensive clinical trial dataset where we shrink towards a power law dose-response relationship, and on monthly influenza and pneumonia data where we shrink moving average lag parameters towards smoothness. Software to implement the proposed approach is provided in the R package SUBSET [15].

The current research included a comparison of several Bayesian shrinkage methods under different loss functions .Results showed that the estimation method is affected by both the sample size, the initial value of the distribution parameter, and the estimation method.

2. Kumaraswamy distribution

The Kumaraswamy distribution is a continuous probability distribution that is widely used in statistics to model random variables bounded between a lower and an upper limit. It was introduced by G. Kumaraswamy in 1980 and is characterized by its flexibility in modelling a variety of shapes, including symmetric, skewed, and unimodal distributions.

The probability density function (PDF) of the Kumaraswamy distribution is given by [2]:

$$f(x,\alpha,\beta) = \alpha\ \beta\ x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$$

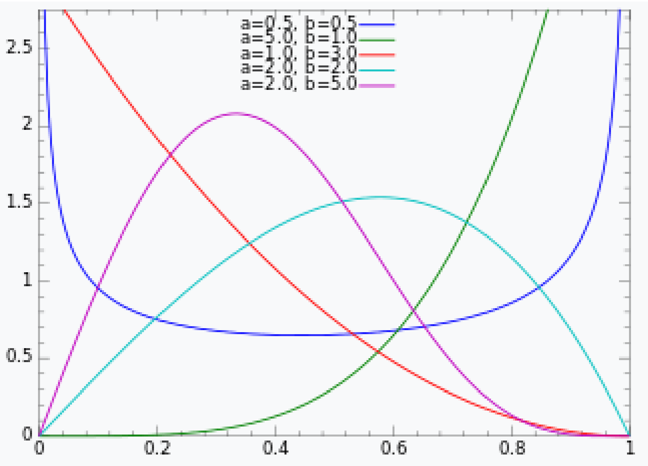
$$with\ \ 0 \le x \le 1$$

$$\alpha,\beta\ > 0$$

And the cumulative density for the distribution can be:-

$$F(x,\alpha,\beta) = 1-(1-x^{\alpha})^{\beta}$$

The following figure represents probability density function with various parameter values



Fig(1) probability density function for Kumaraswamy distribution [6]

The following figure represents cumulative distribution function with various parameter values

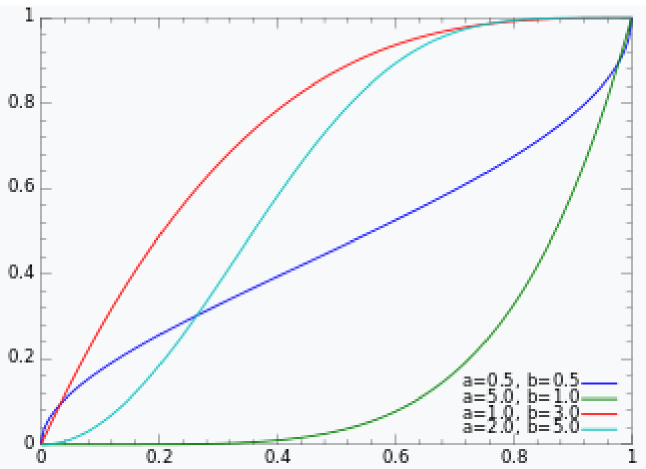


Fig (2) Cumulative distribution function for Kumaraswamy distribution [6]

3. Estimation methods

Statistical estimation is a process of using sample data to estimate parameters of a statistical distribution. There are various methods for statistical estimation such that:

3.1. Maximum Likelihood Estimation Method (MLE)

The likelihood function for the KD with (n) sample size [2]

$$L(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f(x_i, \alpha, \beta)$$

By substituting the formula (1), the function is as follows

$$L(x_1, x_2, \dots, x_n) = (\alpha\beta)^n \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1 - x_i^{\alpha})^{\beta-1}$$

By tacking logarithm we get:

$$Log\big(L(x_1, x_2, \dots, x_n)\big) = nLog(\alpha) + nLog(\beta) + (\alpha - 1)\sum_{i=1}^{n} Log(x_i) + (\beta - 1)\sum_{i=1}^{n} Log(1 - x_i^{\alpha})$$

And taking the partial derivative to $(Log\big(L(x_1, x_2, \dots, x_n)\big))$ for (α) first time and (β) second time and each derivative is equal to zero we get:

$$\hat{\beta}_{mle} = \frac{-n}{\sum_{i=1}^{n} Log(1 - x_i^{\hat{\alpha}_{mle}})}$$

$$\hat{\alpha}_{mle} = -\frac{n}{\sum_{i=1}^{n} Log(x_i) - \big(\hat{\beta}_{mle} - 1\big)\sum_{i=1}^{n} \frac{x_i^{\hat{\alpha}_{mle}}\, Log(\hat{\alpha}_{mle})}{1 - x_i^{\hat{\alpha}_{mle}}}}$$

With $(\hat{\alpha}_{mle}\ and\ \hat{\beta}_{mle})$ represent Maximum Likelihood Estimators for $(\alpha\ and\ \beta)$ respectively. To obtain the estimators from the previous formulas, one of the numerical methods is adopted (Newton Raphson)

3.2. Shrinkage Bayesian estimation method (SBE)

In Bayesian estimation, prior beliefs about parameters are combined with likelihood functions to obtain posterior distributions. Bayes' theorem is used to update prior beliefs based on observed data.

$$f(\beta/n) = \frac{\sum_{i=1}^{n} Log(1-x_i)^n \beta^{n-1} e^{-\beta \sum_{i=1}^{n} Log(1-x_i)}}{\Gamma(n)}$$

There are many loss functions for Shrinkage Bayesian estimation method such that:-

Shrinkage Bayesian estimation refers to a Bayesian approach that incorporates a shrinkage prior to regularize parameter estimates. This regularization helps to mitigate the impact of extreme or influential observations and can improve the stability and accuracy of the estimates.

Shrinkage Bayesian estimator depend on MLE estimation with the following properties [2]. With $(\alpha = 1,2,3)$ the posterior distribution of the parameter (β) is the gamma distribution $(G(n, \sum_{i=1}^{n} Log(1 - x_i))$which has the following probability density function

1-Shrinkage Bayesian estimation under square loss function (SBE-SLF) [4]

$$L_1\big(\beta, \hat{\beta}_s\big) = [\beta - \hat{\beta}_s]^2$$

the posterior risk function (PRF) of (β)can be calculated as

$$\rho_1\big(\beta, \hat{\beta}_{s1}\big) = E_\beta[\beta - \hat{\beta}_{s1}]^2 = \hat{\beta}_{s1}{}^2 - 2\hat{\beta}_{s1}\frac{n}{\sum_{i=1}^{n} Log(1 - x_i)} + \frac{n(n+1)}{[\sum_{i=1}^{n} Log(1 - x_i)]^2}$$

$$\frac{\partial\rho_1\big(\beta, \hat{\beta}_{s1}\big)}{\partial\hat{\beta}_{s1}} = 2\hat{\beta}_{s1}{}^2 - 2\frac{n}{\sum_{i=1}^{n} Log(1 - x_i)}$$

$$\frac{\partial\rho_1(\beta,\hat{\beta}_{s1})}{\partial\hat{\beta}_{s1}}=0$$

the generalized Bayesian estimator of the parameter(β) under the squared error loss function will be

$$\hat{\beta}_{s1}=\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}$$

the shrinkage estimator is defined as

$$\hat{\beta}_{sh1}=\tau(\hat{\beta}_{s1}-\beta_0)+\beta_0$$
$$\rho_1(\beta,\hat{\beta}_{sh1})=E[\beta-\hat{\beta}_{sh1}]^2$$
$$\rho_1(\beta,\hat{\beta}_{sh1})=E_\beta[\tau(\hat{\beta}_{s1}-\beta_0)+\beta_0-\beta]^2$$
$$\rho_1(\beta,\hat{\beta}_{sh1})=[\tau\hat{\beta}_{s1}]^2+(2\tau+2\tau^2)\beta_0\hat{\beta}_{s1}+[1-\tau]^2{\beta_0}^2-2\tau\hat{\beta}_{s1}\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}+2(\tau-1)\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}\beta_0+\frac{n(n+1)}{[\sum_{i=1}^{n}Log(1-x_i)]^2}$$
$$\frac{\partial\rho_1(\beta,\hat{\beta}_{sh1})}{\partial\tau}=2\tau{\beta_{sh1}}^2+2(1-2\tau)\beta_0\hat{\beta}_{s1}-2(1-\tau){\beta_0}^2-\frac{2n(\hat{\beta}_{s1}+\beta_0)}{\sum_{i=1}^{n}Log(1-x_i)}$$
$$\frac{\partial\rho_1(\beta,\hat{\beta}_{sh1})}{\partial\tau}=0$$
$$\tau=\frac{\sum_{i=1}^{n}Log(1-x_i)\,({\beta_0}^2-\beta_0\left(\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}\right)+n(\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}+\beta_0)}{\sum_{i=1}^{n}Log(1-x_i)\,(\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}-\beta_0)^2}$$

The generalized Bayesian shrinkage estimator under the squared error loss function (SBE-SLF) will be:-

$$\hat{\beta}_{sh1}=\left(\frac{\sum_{i=1}^{n}Log(1-x_i)\,({\beta_0}^2-\beta_0\left(\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}\right)+n(\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}+\beta_0)}{\sum_{i=1}^{n}Log(1-x_i)\left(\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}-\beta_0\right)^2}\right)\left(\frac{n}{\sum_{i=1}^{n}Log(1-x_i)}-\beta_0\right)+\beta_0$$

2-Bayesian Shrinkage Estimator under LINEX Loss Function (SBE- LINEXF)

the generalized Bayesian shrinkage estimator (GBSE) under the LINEX loss function can be [14]:-

$$L_2(\Delta)=e^{a\Delta}-(a\Delta+1)$$

With ($\Delta=\frac{\hat{\beta}_{sh2}}{\beta}$)

$$\rho_2(\beta,\hat{\beta}_{s2})=E_b[e^{a\left(\frac{\hat{\beta}_{sh2}}{\beta}-1\right)}-a\left(\frac{\hat{\beta}_{sh2}}{\beta}-1\right)-1]$$

$$\rho_2(\beta,\hat{\beta}_{s2})=e^aE_b\left[e^{a\left(\frac{\hat{\beta}_{sh2}}{\beta}\right)}\right]-a\hat{\beta}_{sh2}E_b\left(\frac{1}{\beta}\right)+a-1$$

With

$$\beta\sim Gamma\ dist.(n,\sum_{i=1}^{n}Log(1-x_i))$$

$\frac{1}{\beta}$ $inv.Gamma\ dist.(n,\sum_{i=1}^{n}Log(1-x_i))$

the pdf will be :-

$$f\left(\frac{1}{\beta}\right)=\frac{[\sum_{i=1}^{n} Log(1-x_i)]^n}{\Gamma(n)}\beta^{-(n+1)}e^{-[\frac{\sum_{i=1}^{n} Log(1-x_i)}{\beta}]}$$

$$E\left(\frac{1}{\beta}\right)=\frac{\sum_{i=1}^{n} Log(1-x_i)}{n-1}$$

$$\rho_2\left(\beta,\hat{\beta}_{s2}\right)=e^{a}\,(\frac{\sum_{i=1}^{n} Log(1-x_i)}{\sum_{i=1}^{n} Log(1-x_i)-a\hat{\beta}_{s2}})^n-a\hat{\beta}_{s2}\left[\frac{\sum_{i=1}^{n} Log(1-x_i)}{n-1}\right]+a-1$$

$$\frac{\partial\rho_2\left(\beta,\hat{\beta}_{s2}\right)}{\partial\hat{\beta}_{s2}}=an[\sum_{i=1}^{n} Log(1-x_i)]^n\,e^{-a}[\sum_{i=1}^{n} Log(1-x_i)-a\hat{\beta}_{s2}]^{-(n+1)}-\frac{a[\sum_{i=1}^{n} Log(1-x_i)]}{n-1}$$

$$\frac{\partial\rho_2\left(\beta,\hat{\beta}_{s2}\right)}{\partial\hat{\beta}_{s2}}=0$$

Then generalized Bayesian estimator under LINEX loss function will be :-

$$\hat{\beta}_{s2}=\frac{1}{a}(\sum_{i=1}^{n} Log(1-x_i)-(n\sum_{i=1}^{n} Log(1-x_i)^{n-1}\,e^{-a}))^{\frac{1}{n+1}}$$

The Bayesian Shrinkage Estimator under LINEX Loss Function will be :-

$$\rho_2\left(\beta,\hat{\beta}_{sh2}\right)=E_b\left(e^{a\left(\frac{\hat{\beta}_{sh2}}{\beta}-1\right)}\right)-a\hat{\beta}_{sh2}E_b\left(\frac{1}{\beta}\right)+a-1$$

$$\rho_2\left(\beta,\hat{\beta}_{sh2}\right)=e^{-a}(\frac{\sum_{i=1}^{n} Log(1-x_i)}{\sum_{i=1}^{n} Log(1-x_i)-a\hat{\beta}_{sh2}})^n-a\hat{\beta}_{sh2}\frac{\sum_{i=1}^{n} Log(1-x_i)}{n-1}+a-1$$

the shrinkage estimator is defined as

$$\hat{\beta}_{sh2}=\tau\left(\hat{\beta}_{s2}-\beta_0\right)+\beta_0$$

$$\rho_2\left(\beta,\hat{\beta}_{sh2}\right)=\frac{e^{-a}(\sum_{i=1}^{n} Log(1-x_i))^n}{[\sum_{i=1}^{n} Log(1-x_i)-a\tau\left(\hat{\beta}_{s2}-\beta\right)-a\beta_0]^n}-a\frac{\sum_{i=1}^{n} Log(1-x_i)}{n-1}\tau\left(\hat{\beta}_{s2}-\beta\right)-a\beta_0\frac{\sum_{i=1}^{n} Log(1-x_i)}{n-1}$$

$$\frac{\partial\rho_2\left(\beta,\hat{\beta}_{sh2}\right)}{\partial\tau}=\frac{an\left(\hat{\beta}_{s2}-\beta\right)e^{-a}(\sum_{i=1}^{n} Log(1-x_i))^n}{[\sum_{i=1}^{n} Log(1-x_i)-a\tau\left(\hat{\beta}_{s2}-\beta\right)-a\beta_0]^{(n+1)}}-a\frac{\sum_{i=1}^{n} Log(1-x_i)\left(\hat{\beta}_{s2}-\beta\right)}{n-1}$$

$$\frac{\partial\rho_2\left(\beta,\hat{\beta}_{sh2}\right)}{\partial\tau}=0$$

We get

$$\tau=\frac{[\sum_{i=1}^{n} Log(1-x_i)-a\beta_0]-[n(n-1)\sum_{i=1}^{n} Log(1-x_i)^{n-1}\,e^{-a}]^{\frac{1}{n+1}}}{a[\frac{1}{a}[\sum_{i=1}^{n} Log(1-x_i)-\left[n\sum_{i=1}^{n} Log(1-x_i)^{n-1}\,e^{-a}\right]^{\frac{1}{n+1}}-\ \beta_0]}$$

$$\hat{\beta}_{sh2}=\left(\frac{[\sum_{i=1}^{n} Log(1-x_i)-a\beta_0]-\left[n(n-1)\sum_{i=1}^{n} Log(1-x_i)^{n-1}\,e^{-a}\right]^{\frac{1}{n+1}}}{a[\frac{1}{a}[\sum_{i=1}^{n} Log(1-x_i)-\left[n\sum_{i=1}^{n} Log(1-x_i)^{n-1}\,e^{-a}\right]^{\frac{1}{n+1}}-\ \beta_0]}\right)\left(\frac{1}{a}\left[\sum_{i=1}^{n} Log(1-x_i)-\left[n\sum_{i=1}^{n} Log(1-x_i)^{n-1}\,e^{-a}\right]^{\frac{1}{n+1}}\right]-\beta_0\right)+\beta_0$$

4. Generate Kumaraswamy Distribution

To generate random sample with (α, β) by tacking the following cumulative distribution function

$F(x, \alpha, \beta) = 1 - (1 - x^{\alpha})^{\beta}$

Let $(R = F(x, \alpha, \beta))$

With (R) represent random sample within [0-1]

$R = 1 - (1 - x^{\alpha})^{\beta}$

$x = [e^{\frac{\log(1-R)}{\beta}}]^{[\frac{1}{\alpha}]}$

$x = [1 - e^{\frac{\log(1-R)}{\beta}}]^{[\frac{1}{\alpha}]}$

5. Experimental results

In order to compare the estimation methods with various random samples, initial distribution parameters and estimation methods, number of simulation experiments with A sufficient iterations was done Comparing estimators and mean square error for the estimation methods was done by [7]

$\emptyset_j = \min(|\theta - \hat{\theta}_j|)$

Such that

$\emptyset_j$ Represent minimum absolute of deference for estimation method (j)

θ Represent true value

$\hat{\theta}_j$ Represent estimator value for estimation method (j)

$$Mse = \frac{\sum_{it=1}^{rep}[\hat{\theta}_j - \theta]^2}{rep}$$

Such that

Mse Represent mean square error

The following tables and figures represent the experimental results

Table(1) shows the best estimation method based on minimum absolute of deference

β	α	n	$\hat{\alpha}_{\text{MLE}}$	$\hat{\alpha}_{\text{SBE-SLF}}$	$\hat{\alpha}_{\text{SBE- LINEXF}}$	$\emptyset$	i
0.25	0.5	15	0.773529	0.23427	0.801411	1.57E-02	2
		50	0.246427	0.249514	0.246039	4.86E-04	2
		100	0.250155	0.250478	0.250213	1.55E-04	1
0.25	1	15	0.309897	0.263672	0.233098	1.37E-02	2
		50	0.247413	0.246969	0.248007	1.99E-03	3
		100	0.248387	0.250311	0.247908	3.11E-04	2
0.25	1.5	15	0.337232	0.265057	0.25672	6.72E-03	3
		50	0.336797	0.24593	0.341097	4.07E-03	2
		100	0.251945	0.249909	0.252395	9.07E-05	2
0.5	0.5	15	0.576319	0.481356	0.591162	1.86E-02	2
		50	0.496413	0.499452	0.49867	5.48E-04	2
		100	0.512308	0.49994	0.512191	5.99E-05	2
0.5	1	15	0.435081	0.515245	0.411901	1.52E-02	2
		50	0.491155	0.503602	0.489607	3.60E-03	2
		100	0.498275	0.500097	0.498632	9.71E-05	2
0.5	1.5	15	0.580816	0.489582	0.55214	1.04E-02	2
		50	0.491308	0.498348	0.49284	1.65E-03	2
		100	0.495627	0.499592	0.495325	4.08E-04	2
0.75	0.5	15	0.721992	0.805406	0.724121	2.59E-02	3
		50	0.741347	0.750704	0.747152	7.04E-04	2
		100	0.749482	0.749817	0.749908	9.17E-05	3
0.75	1	15	0.89237	0.736032	0.868202	1.40E-02	2
		50	0.742496	0.749014	0.744016	9.86E-04	2
		100	0.858245	0.749812	0.858188	1.88E-04	2
0.75	1.5	15	0.724582	0.775049	0.735764	1.42E-02	3
		50	0.737846	0.75587	0.744206	5.79E-03	3
		100	0.816631	0.74984	0.816333	1.60E-04	2

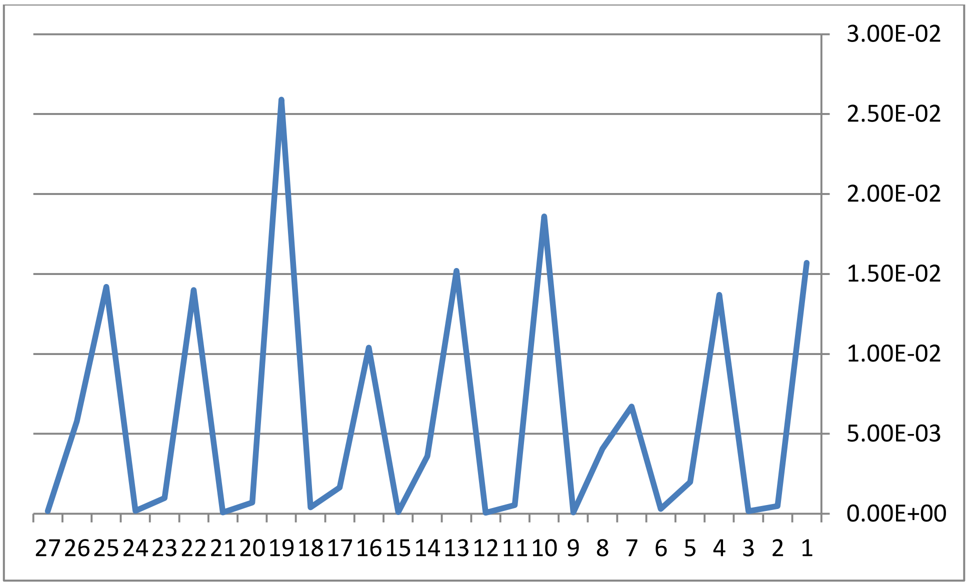


Figure (3) shows the best values for the error criterion minimum absolute of deference

The previous table and figure showed the effect of the estimation method on both the sample size and the parameter values of the distribution. We notice, according to this measure, that the first estimation method was better one time, while the second estimation method was better by 20 times, while the third method was better by 6 times.

Table(2) shows the best estimation method based on mean square error

β	α	n	MSE_{MLE}	$MSE_{\text{SBE-SLF}}$	$MSE_{\text{SBE- LINEXF}}$	$Best$	i
0.25	0.5	15	0.607863	4.21E-04	0.651213	4.21E-04	2
		50	2.86E-05	8.37E-07	1.57E-05	8.37E-07	2
		100	3.15E-08	2.35E-07	5.26E-08	3.15E-08	1
0.25	1	15	1.41E-02	5.16E-04	2.94E-02	5.16E-04	2
		50	3.43E-05	1.13E-05	8.37E-06	8.37E-06	3
		100	2.69E-06	1.02E-07	4.84E-06	1.02E-07	2
0.25	1.5	15	7.96E-03	4.26E-04	5.94E-05	5.94E-05	3
		50	7.53E-03	3.00E-05	8.32E-03	3.00E-05	2
		100	2.20E-05	1.91E-07	2.60E-05	1.91E-07	2
0.5	0.5	15	1.27E-02	2.09E-03	1.14E-02	2.09E-03	2
		50	1.58E-05	4.53E-07	2.87E-06	4.53E-07	2
		100	3.06E-04	1.97E-08	3.04E-04	1.97E-08	2
0.5	1	15	7.51E-03	6.98E-04	1.82E-02	6.98E-04	2
		50	2.15E-04	5.05E-05	3.60E-04	5.05E-05	2
		100	4.79E-06	7.94E-08	3.85E-06	7.94E-08	2
0.5	1.5	15	5.45E-02	1.42E-04	2.23E-02	1.42E-04	2
		50	1.23E-04	3.92E-06	6.87E-05	3.92E-06	2
		100	3.64E-05	2.32E-07	3.89E-05	2.32E-07	2
0.75	0.5	15	1.15E-03	4.06E-03	1.04E-03	1.04E-03	3
		50	7.58E-05	5.09E-07	1.38E-05	5.09E-07	2
		100	4.43E-07	8.68E-08	1.37E-07	8.68E-08	2
0.75	1	15	2.73E-02	1.99E-04	1.76E-02	1.99E-04	2
		50	6.28E-05	2.90E-06	3.58E-05	2.90E-06	2
		100	1.20E-02	5.48E-07	1.20E-02	5.48E-07	2
0.75	1.5	15	6.64E-04	7.05E-04	7.89E-03	6.64E-04	1
		50	1.56E-04	4.76E-05	5.17E-05	4.76E-05	2
		100	4.67E-03	8.97E-08	4.63E-03	8.97E-08	2

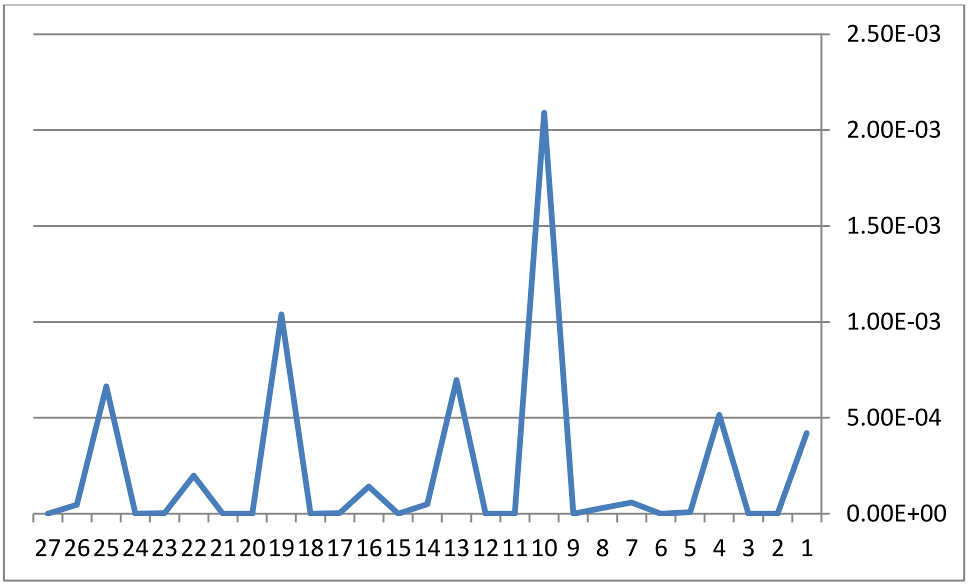


Figure (4) shows the best values for the error criterion mean square error

The previous table and figure showed the effect of the estimation method on both the sample size and the parameter values of the distribution. We notice, according to this measure, that the first estimation method was better 2 times, while the second estimation method was better by 22 times, while the third method was better by 3 times.

6. Conclusions and recommendations

After analyzing the results of various simulation experiments, several conclusions and recommendations emerged

1. The estimation method is affected by the sample size
2. The estimation method is affected by the initial values of the distribution parameters
3. Depending on the (minimum absolute of deference) criterion, the best estimation method is the second, with a percentage of 74%
4. Depending on the (mean square error) criterion, the best estimation method is the second, with a percentage of 82%
5. Calculating estimators for estimation methods in the presence of outliers
6. Comparing nonparametric estimation methods with parametric estimation methods for the assumed distribution.

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